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Risk Aversion: Differential Conditions for the Iso-Utility Curves with Positive Slope in Transformed Two-Parameter Distributions*

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Abstract

The condition of Risk Aversion implies that the Utility Function must be concave. We take into account the dependence of the Utility Function on the return that has any type of two-parameter distribution; it is possible to define Risk and Target, the first one may be the Standard Deviation of the return and the last one usually is the Expected value of the return, as a generic function of these two parameters. Considering the 3D space of *Risk*, *Target* and *Expected Utility*, this paper determines the Differential Conditions for these three functions so that the Expected Utility Function depends decreasingly on Risk and increasingly on Target, that means the iso-utility curves have positive slope in the plane of Risk and Target. As a particular case, we discuss these conditions in the case of the CRRA Utility Function and the Truncated Normal distribution. Furthermore, different measures of Risk are chosen, as Value at Risk (VaR) and Expected Shortfall (ES), to verify if these measures maintain a positive slope of the Iso-utility curves in the Risk-Target plane.

Keywords: Concavity, Differential Conditions, Expected Shortfall, Risk Aversion, Standard Deviation, Value at Risk.

JEL: G11, G14, G23, G24.

1. Introduction

Risk aversion is referred to as the amount an agent is willing to pay in order to avoid risk. In the expected utility theory, the risk aversion measure is generally given by the Arrow-Pratt index, which requires the von Neumann-Morgenstern utility function.

There is no doubt that risk aversion is linked to the concavity of the utility function. For example, the Arrow–Pratt measure of absolute risk-aversion (ARA) relates the degree of concavity of a utility function measured by the curvature index known as the coefficient of absolute risk aversion. As underlined by Machina (1987), since someone with a concave utility function will always prefer receiving the expected value of a gamble to the gamble itself, concave utility functions are termed risk averse.

Machina affirms that in the case of non-expected utility function we can use calculus to extend the results obtained from the expected utility function. In particular he takes into account the concavity in the consequences of the partial derivatives with respect to probabilities of the preference function. Other authors criticize the results obtained by this extension. For example, Montesano (1991) argues that, unlike what happens in the expected utility function, in non-expected utility function we can find examples of agents that prefer the lottery to its expected value (denoting risk attraction) while they prefer a smaller risk and vice versa. In this case, the concavity of the derivatives of the utility function cannot be considered an index of risk aversion for smaller risks.

Li Calzi and Sorato (2004), starting from the consideration that the existing parameterizations of prospect theory are not satisfactory, suggest a parameterization for utility and value functions that works across both the expected utility and prospect theory. With this parameterization the consequent family of functions are twice differentiable and are restricted to have only possible shapes: convex, concave, S-shaped and reverse S-shaped.

The drawback of the suggested parameterization is that the family includes utility (or value) functions which have no representation in closed-form, even though their first derivatives always admit an explicit representation.

We have mentioned some articles that discuss the concavity and the risk aversion by considering properties of the functions in two-dimensional space.

In three-dimensional space, we can quote Lajeri and Nielsen (1998) whose aim is to determine whether one decision maker is more risk averse than another. For this purpose, Lajeri and Nielsen limit themselves to the two-parameter family of random variables and the risk aversion is measured considering the expected utility as a function of mean and standard deviation. In their analysis the concavity of the utility function plays an important role in determining the decision maker's attitude, measured by the marginal rate of substitution between mean and standard deviation, that is, by the slope of an indifference curve. The authors establish also the equivalence of the concept of decreasing absolute prudence (DAP), introduced by Kimball (1990), and the decreasing of the slope of the indifference curves of the utility function. Eichener and Wagener (2001) show that this latter result cannot be generalized for distributions other than the normal distribution.

In their papers, Rothschild and Stiglitz (1970, 1971), propose a partial ordering of probability distributions related to two parameters and criticizes the conventional mean-variance approach because it "gives rise to a complete ordering of distributions (with the same expected value)… The answers of mean-variance analysis are spurious; they hold only if the utility function or the class of distributions is arbitrarily restricted. Furthermore, mean-variance analysis does not seem to provide clues as to what restrictions must be imposed if its results are to hold".

Following this criticism in our paper we consider a general version of the mean-variance approach where the return has a distribution which depends on two parameters, so Expected Returns (that represents the mean in the mean-variance approach), Risk (that may be identified with the Standard Deviation) and Expected Utility Function are functions of two parameters. The question we solve is what are the conditions to be imposed on this Expected Utility Function to preserve the risk aversion, using, as unique restriction, the class in which the distributions are the function of only two parameters.

The result we obtain is that we do not need to refer directly to the concavity of the Expected Utility Function, but more generally, we find the set of the differential conditions so that the iso-utility curves have a positive slope in the plane of Risk–Expected Return. However, these conditions are necessary in order to remain in the condition of risk aversion, ie in the domain where the utility function is concave.

In our paper, we consider a *risk-averse Utility Function U(W)*, where the Wealth is defined as $W = W_0(1+r)$, *r* is the *return* with a generic distribution which depends on two parameters.

The *risk-averse* conditions are related to the first and the second derivatives of the *U(W)* and the degree of risk aversion can be measured by the curvature of the *U(W)*.

We consider now the Expected Utility Function, $\psi = E[U\big((W)\big]$, that is a function of the two parameters. We define also the two functions *Risk* and *Target. Target* is e.g. the Expected value of *r* and *Risk* is e.g. the Standard Deviation of *r. Risk* and *Target* are also function of the two parameters, and we can consider the three dimensions space $[Risk, Target, \psi]$, where we have the implicit function $Target_{\psi}(Risk)$, defined by the intercept of $\psi(Risk, Target)$ with a generic horizontal plane.

This paper determines the Differential Conditions for $Risk, Target \;\; and\, \psi$ so that ψ depends decreasingly on $Risk$ and increasingly on $Target$, that is, the first derivative of the Implicit Function $Target_{\psi}(Risk)$ is positive and, consequently, the Iso-utility curves in the plane of Risk and Target have a positive slope.

As a particular case, the paper describes the *Constant Relative Risk Aversion Utility Function* (*CRRA*) applied to a return that has a Truncated Normal distribution.

The paper is organized as follows. Section 2 introduces the properties for the *Utility Function* when wealth depends on the return *r* considered as a Normal variable. These properties are extended when the return *r* has a generic distribution which depends on two parameters and the definitions of *Risk* and *Target* are transformations of these two parameters. Section 3 defines the Differential Conditions that must be respected when we consider a parametric representation of the surface concerning the *Risk*, *Target* and ψ and we desire that depends decreasingly on $Risk$ and increasingly on $Target$, i.e. the iso-utility curves in the plane of Risk and Target have a positive slope. The conditions concern any two-parameter distribution. This is obtained without restrictions for the *U(W)* or definitions of

Risk and *Target*. Section 4 takes into consideration the *CRRA Utility Function* and the Truncated Normal variable for the return. Using its Expected value for *Target*, the Standard Deviation, VaR and Expected Shortfall of the return are analyzed as measure of risk. Only the Standard Deviation respects the Differential Conditions and has the iso-utility curves with a positive slope. Section 5 contains the conclusions.

2. Utility Function in the Case of Normal Distribution

Let us consider the *Utility Function* $U(W)$, where W is the wealth (or a quantity of the uncertain payment), given by:

(2.1) $$W = W_0(1 + r),$$

with the initial value W_0 and the return r.

If $U(W)$ represents a *risk-averse* person with insatiable appetite:

(2.2) $$U'(W) > 0\,; \qquad U''(W) < 0$$

(2.3) $$ARA = Absolute\ Risk\ Aversion = -\frac{U''(W)}{U'(W)} > 0$$

Theorem 2.1*: Let $\succcurlyeq$ be an expected utility preference relation on all normal distributions $N(\mu, \sigma^2)$ for the return r. Then there exists a mean-variance Expected Utility Function $\psi(\sigma, \mu)$ which describes $\succcurlyeq$.*

In the case of risk aversion, $\psi(\sigma, \mu)$ has the following partial derivatives and the first derivative of the implicit function $\mu_\psi(\sigma)$:

(2.4) $$\frac{\partial\psi(\sigma,\mu)}{\partial\mu} > 0, \quad \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} < 0, \qquad \Rightarrow \frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\frac{\partial\psi(\sigma,\mu)}{\partial\mu}} > 0$$

Proof: *Appendix A.* □

Theorem 2.1 describes a reasonable and intuitive behavior for the *risk-averse* investor translated in 3 dimensions space $[\sigma, \mu, \psi(\sigma, \mu)]$ when $r \sim N(\mu, \sigma^2)$.

More generally we consider the return $r \sim G(\sigma, \mu)$, where *G* is any two-parameter distribution and $g(r, \sigma, \mu)$ is the probability density function defined for $r \subseteq [\delta_1, \delta_2]$.

It is possible to compute the following *Expected Utility Function*, $\psi(\sigma, \mu)$.

(2.5) $$\psi(\sigma, \mu) \equiv E[U(W)] = E[U(1 + r)] = \int_{\delta_1}^{\delta_2} U\,(1 + r) g(r, \sigma, \mu) dr$$

The *Target* can be defined, as usual, as the *Expected value* of *r* :

$$Target = T(\sigma, \mu) = \int_{\delta_1}^{\delta_2} r\, g(r, \sigma, \mu) dr$$

and *Risk*, e.g., as the *Standard Deviation* of *r* :

$$Risk = R(\sigma, \mu) = \sqrt{\int_{\delta_1}^{\delta_2} [r - T(\sigma, \mu)]^2\, g(r, \sigma, \mu) dr}$$

We can choose any other definition for *Risk* as a generic functions of (σ, μ), e.g. *VaR* or *Expected Shortfall (ES)*. In the same line it is also possible to introduce a generic transformation to define the *Target :*

$$Risk = R(\sigma, \mu)$$

$$Target = T(\sigma, \mu)$$

where $R(\sigma, \mu)$ *and* $T(\sigma, \mu)$ are generic functions of (σ, μ) and we assume that they are at least once differentiable with continuous first derivatives.

For sake of simplicity we named the generic parameters as (σ, μ); later we will introduce the specific case of the Truncated Normal variable, and this choice allows us not to rename the parameters.

Considering a *risk-averse Utility Function* we want to determine which conditions must be satisfied by the three functions $R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)$ so that in the parametric space $[R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)]$ the following conditions are true :

$$\begin{cases} \dfrac{\partial\psi}{\partial R} < 0 \\ \dfrac{\partial\psi}{\partial T} > 0 \end{cases} \Rightarrow \frac{dT_\psi(R)}{dR} = -\frac{\frac{\partial\psi}{\partial R}}{\frac{\partial\psi}{\partial T}} > 0 \tag{2.6}$$

that is the first derivatives of the Implicit Function $T_\psi(R)$, defined by the intercept of $\psi(R,T)$ with a generic horizontal plane, is positive. The (2.6) means the Iso-utility curves in the plane $[R(\sigma,\mu), T(\sigma,\mu)]$ have positive slope.

Is it sufficient that *U(W)* is *risk-averse* or is it necessary to introduce others conditions for the three functions mentioned above? In the following section we give the answer.

3. Differential Conditions for the Concavity of the Expected Utility Functions: The specific Case of the Truncated Normal

As already introduced in Section 2, we consider $r \sim G(\sigma,\mu)$ and define *Risk* and *Target* as a functions of (σ,μ):

$$Risk = R(\sigma,\mu), \quad Target = T(\sigma,\mu) \tag{3.1}$$

The *Expected Utility Function* $\psi(\sigma,\mu)$ is defined in (2.5).

Now we want to find the conditions for *Risk* and *Target* so that the (2.6) is satisfied. First of all we have to impose the condition that the transformation $[\sigma,\mu] \rightarrow [R(\sigma,\mu), T(\sigma,\mu)]$ defined by (3.1) is bijective.

This condition implies that the determinant of the Jacobian matrix J must be different from zero:

$$detJ = det\begin{pmatrix} \frac{\partial R(\sigma,\mu)}{\partial \sigma} & \frac{\partial R(\sigma,\mu)}{\partial \mu} \\ \frac{\partial T(\sigma,\mu)}{\partial \sigma} & \frac{\partial T(\sigma,\mu)}{\partial \mu} \end{pmatrix} \neq 0 \tag{3.2}$$

Consider a parametric representation of a surface, where:

x axis = *Risk* = $R(\sigma,\mu)$.
y axis = *Target* = $T(\sigma,\mu)$.
z axis = *Expected Utility Function* = $\psi(\sigma,\mu)$.

This surface is described in the space $[R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)]$ by the three functions $R(\sigma,\mu)$, $T(\sigma,\mu)$, $\psi(\sigma,\mu)$ that depends on (σ,μ) defined in $[(\sigma_{Min},\sigma_{Max}) \times (\mu_{Min},\mu_{Max})]$ in the cartesian space (σ,μ).

Using the vector notation, the surface is defined by vector $\boldsymbol{s}(\sigma,\mu)$ in the space:

$$[R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)],$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the relative unit vectors:

$$\boldsymbol{s}(\sigma,\mu) = R(\sigma,\mu)\boldsymbol{i} + T(\sigma,\mu)\boldsymbol{j} + \psi(\sigma,\mu)\boldsymbol{k} \tag{3.3}$$

For regularity of the surface, the Jacobian Matrix J_1:

$$J_1 = \begin{pmatrix} \frac{\partial R(\sigma,\mu)}{\partial \sigma} & \frac{\partial R(\sigma,\mu)}{\partial \mu} \\ \frac{\partial T(\sigma,\mu)}{\partial \sigma} & \frac{\partial T(\sigma,\mu)}{\partial \mu} \\ \frac{\partial \psi(\sigma,\mu)}{\partial \sigma} & \frac{\partial \psi(\sigma,\mu)}{\partial \mu} \end{pmatrix} \tag{3.4}$$

must have rank two; e.g. this condition is satisfied if (3.2) is true.

The orthogonal unit vector of the surfaces is done by:

$$\frac{\frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial\sigma}\times\frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial\mu}}{\left\|\frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial\sigma}\times\frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial\mu}\right\|}$$

where:

$$(3.5)\quad \frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial\sigma}\times\frac{\partial \boldsymbol{s}(\sigma,\mu)}{\partial\mu}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial R(\sigma,\mu)}{\partial\sigma} & \frac{\partial T(\sigma,\mu)}{\partial\sigma} & \frac{\partial \psi(\sigma,\mu)}{\partial\sigma} \\ \frac{\partial R(\sigma,\mu)}{\partial\mu} & \frac{\partial T(\sigma,\mu)}{\partial\mu} & \frac{\partial \psi(\sigma,\mu)}{\partial\mu} \end{vmatrix}$$

$$=\left[\frac{\partial T}{\partial\sigma}\frac{\partial\psi}{\partial\mu}-\frac{\partial\psi}{\partial\sigma}\frac{\partial T}{\partial\mu}\right]\boldsymbol{i}-\left[\frac{\partial R}{\partial\sigma}\frac{\partial\psi}{\partial\mu}-\frac{\partial\psi}{\partial\sigma}\frac{\partial R}{\partial\mu}\right]\boldsymbol{j}+\left[\frac{\partial R}{\partial\sigma}\frac{\partial T}{\partial\mu}-\frac{\partial T}{\partial\sigma}\frac{\partial R}{\partial\mu}\right]\boldsymbol{k}$$

and the dependence on (σ,μ) is omitted in the last formula.

We can see now some examples. In the following graphs the red arrows are the orthogonal unit vectors, in 3D and 2D respectively.

In both the Examples of Figure 3.1a and Figure 3.1b the Iso-utility curves have positive slope. The projection of the normal vector on the *[R, T]* plane has positive component along *R-axis* and negative component along *T- axis*.

In Figure 3.1c in some parts the Iso-utility curves have negative slope, and the projection of the normal vector on the *[R, T]* plane has negative component along *R-axis* and negative component along *T-axis*.

Figure 3.1a: Example 1

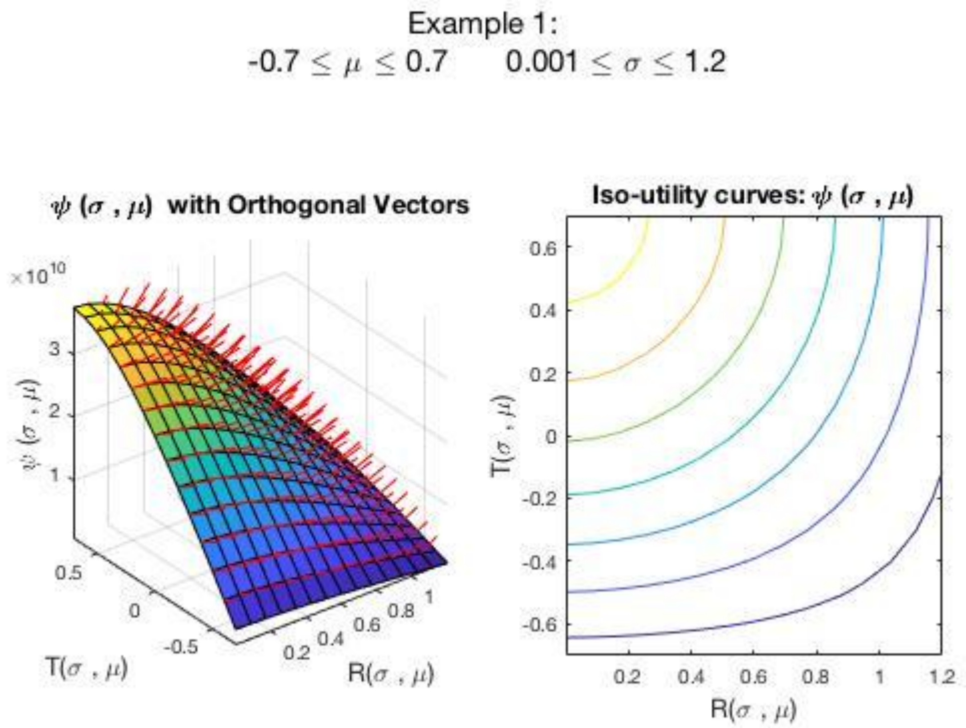


Figure 3.1b: Example 2

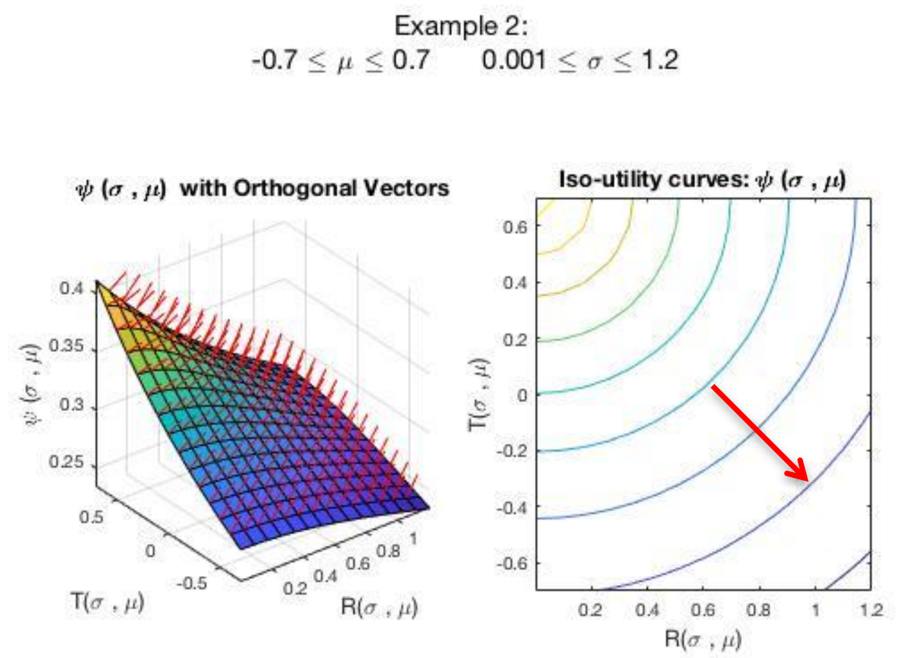


Figure 3.1c: Example 3

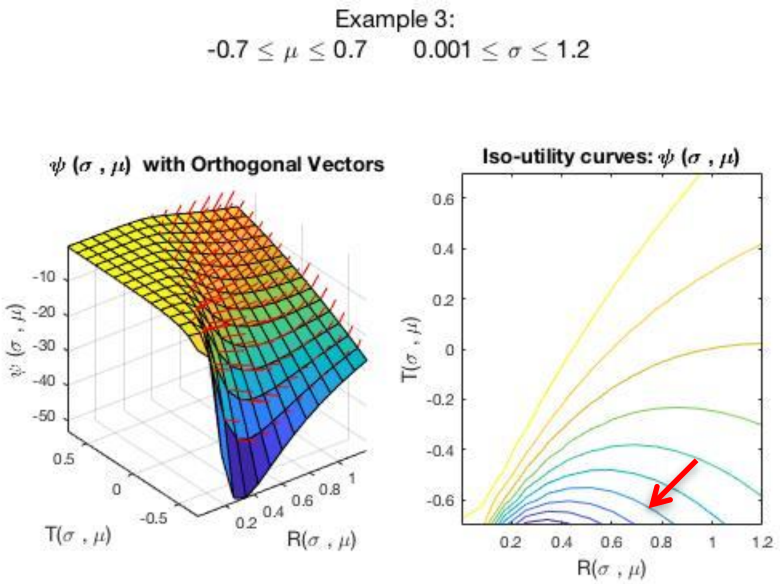
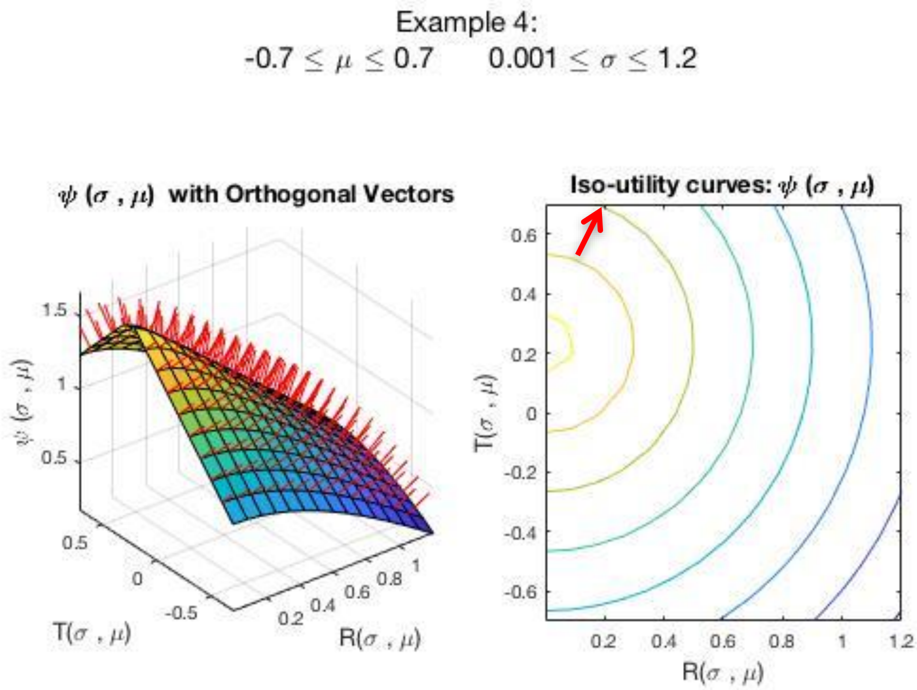


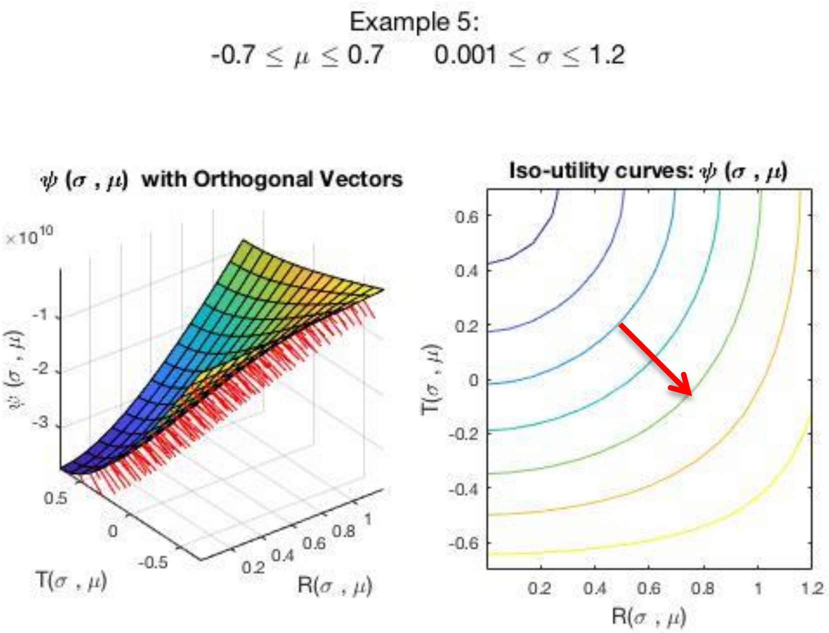
Figure 3.1d: Example 4



In Figure 3.1d in some parts the Iso-utility curves have negative slope, and the projection of the unit vector in the in the *[R, T]* plane has positive component along *R-axis* and positive component along *T-axis*.

It is clear that we wish positive component along *R-axis* and negative component along *T-axis*; but this is not sufficient. As a further condition we need the positiveness of the component along the $\psi - axis$. In fact, if this component is negative, despite the iso-utility curves can maintain the positive slope, for the same value of *R* we can have greater utility in correspondence of a value T_1 lower than a value T_2. If we consider as *Target* the Expected Return and as *Risk* the Standard Deviation, we would be in the situation that for the same value of Standard Deviation we have more utility with lower Expected Return than with a greater Expected Return .

Figure 3.1e: Example 5



In the Figure 3.1e, where the component along $\psi - axis$ is negative, the Iso-utility curves have positive slope, but the yellow ones have greater utility than the blue ones, even if for the same value of *Risk* the *Target* is lower: for the same $Risk \approx 0.2$, we have $Target \approx -0.6$ for the yellow Iso-utility curve, lower than $Target \approx 0.4$ for the blue Iso-utility.

In short, the Iso-utility curves have positive slope and we avoid the situation of Figures 3.1c, 3.1d and 3.1e if the components of the orthogonal unit vectors are positive for $R - axis$ and $\psi - axis$ and negative for $T - axis$ as in the Figures 3.1a and 3.1b.

These considerations lead to the following Differential Conditions:

(3.6)
$$\begin{cases} \frac{\partial T}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial T}{\partial \mu} > 0 & : Differential\ Condition\ 1 \equiv DC1 \\ \frac{\partial R}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial R}{\partial \mu} > 0 & : Differential\ Condition\ 2 \equiv DC2 \\ \frac{\partial R}{\partial \sigma}\frac{\partial T}{\partial \mu} - \frac{\partial T}{\partial \sigma}\frac{\partial R}{\partial \mu} > 0 & : Differential\ Condition\ 3 \equiv DC3 \end{cases}$$

Note that *DC3* is the same as in the expression (3.2).

The condition (3.2), for which the transformation $[\sigma, \mu] \rightarrow [R, T]$ is bijective, implies that the inverse transformation $\sigma(R,T)\,, \mu(R,T)$ exists locally:

$$\psi(\sigma, \mu) = \psi\big(\sigma(R,T)\,, \mu(R,T)\big) = \psi(R,T)$$

Computing the partial derivatives:

(3.7)
$$\frac{\partial \psi\big(\sigma(R,T)\,, \mu(R,T)\big)}{\partial R} = \frac{\partial \psi}{\partial \sigma}\frac{\partial \sigma}{\partial R} + \frac{\partial \psi}{\partial \mu}\frac{\partial \mu}{\partial R}$$

$$\frac{\partial \psi\big(\sigma(R,T)\,, \mu(R,T)\big)}{\partial T} = \frac{\partial \psi}{\partial \sigma}\frac{\partial \sigma}{\partial T} + \frac{\partial \psi}{\partial \mu}\frac{\partial \mu}{\partial T}$$

By the Theorem of the Inverse Function, we have:

$$\begin{pmatrix} \frac{\partial\sigma(R,T)}{\partial R} & \frac{\partial\sigma(R,T)}{\partial T} \\ \frac{\partial\mu(R,T)}{\partial R} & \frac{\partial\mu(R,T)}{\partial T} \end{pmatrix} = \begin{pmatrix} \frac{\partial R(\sigma,\mu)}{\partial\sigma} & \frac{\partial R(\sigma,\mu)}{\partial\mu} \\ \frac{\partial T(\sigma,\mu)}{\partial\sigma} & \frac{\partial T(\sigma,\mu)}{\partial\mu} \end{pmatrix}^{-1}$$

that has solution for the condition (3.2). We can write:

$$\begin{pmatrix} \frac{\partial\sigma(R,T)}{\partial R} & \frac{\partial\sigma(R,T)}{\partial T} \\ \frac{\partial\mu(R,T)}{\partial R} & \frac{\partial\mu(R,T)}{\partial T} \end{pmatrix} = \begin{pmatrix} \frac{1}{detJ}\frac{\partial T}{\partial\mu} & -\frac{1}{detJ}\frac{\partial R}{\partial\mu} \\ -\frac{1}{detJ}\frac{\partial T}{\partial\sigma} & \frac{1}{detJ}\frac{\partial R}{\partial\sigma} \end{pmatrix}$$

and substituting in (3.7) we have:

$$\frac{\partial\psi}{\partial R} = \frac{1}{detJ}\frac{\partial\psi}{\partial\sigma}\frac{\partial T}{\partial\mu} - \frac{1}{detJ}\frac{\partial\psi}{\partial\mu}\frac{\partial T}{\partial\sigma} \quad \Rightarrow \ -detJ\frac{\partial\psi}{\partial R} = \frac{\partial T}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial T}{\partial\mu}$$

$$\frac{\partial\psi}{\partial T} = -\frac{1}{detJ}\frac{\partial\psi}{\partial\sigma}\frac{\partial R}{\partial\mu} + \frac{1}{detJ}\frac{\partial\psi}{\partial\mu}\frac{\partial R}{\partial\sigma} \ \Rightarrow \quad detJ\frac{\partial\psi}{\partial T} = \frac{\partial R}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial R}{\partial\mu}$$

Substituting in (3.6) we obtain:

$$(3.8)\ \begin{cases} \frac{\partial T}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial T}{\partial\mu} > 0 \\ \frac{\partial R}{\partial\sigma}\frac{\partial\psi}{\partial\mu} - \frac{\partial\psi}{\partial\sigma}\frac{\partial R}{\partial\mu} > 0 \\ \frac{\partial R}{\partial\sigma}\frac{\partial T}{\partial\mu} - \frac{\partial T}{\partial\sigma}\frac{\partial R}{\partial\mu} > 0 \end{cases} \Rightarrow \begin{cases} -detJ\frac{\partial\psi}{\partial R} > 0 \\ detJ\frac{\partial\psi}{\partial T} > 0 \\ detJ > 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial\psi}{\partial R} < 0 \\ \frac{\partial\psi}{\partial T} > 0 \\ detJ > 0 \end{cases} \Rightarrow \begin{cases} \frac{dT_{\psi}(R)}{dR} = -\frac{\frac{\partial\psi}{\partial R}}{\frac{\partial\psi}{\partial T}} > 0 \\ detJ > 0 \end{cases}$$

The inequalities in (3.8) shed light on the meaning of the Differential Conditions in (3.6): the *Expected Utility Function* $\psi(\sigma,\mu)$ depends decreasingly on R and increasingly on T, due to the sign of the *DC3*, that is the sign of the unit vector component along the $\psi - axis$.

The conditions $\partial\psi/\partial R < 0$ and $\partial\psi/\partial T > 0$ are not verifiable in closed form; they are a consequence of (3.6) and they imply that the first derivative of the Implicit Function $T_{\psi}(R)$, defined by the intercept of $\psi(R,T)$ with a generic horizontal plane, is positive.

The inequalities (3.8) generalize the conditions given in *Theorem 2.1* for the Normal distribution because they apply to any two-parameter distribution and to any definition of *Risk* and *Target*.

It is possible to rewrite the (3.8) to determine a geometric explanation. We use, e.g., the hypothesis:

$$\frac{\partial R}{\partial \sigma} > 0; \quad \frac{\partial R}{\partial \mu} < 0; \quad \frac{\partial T}{\partial \sigma} < 0; \quad \frac{\partial T}{\partial \mu} > 0; \quad \frac{\partial \psi}{\partial \mu} > 0; \quad \frac{\partial \psi}{\partial \sigma} < 0 \tag{3.9}$$

From the first Differential Condition we have:

$$DC1 = \frac{\partial T}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial T}{\partial \mu} > 0 \implies -\frac{\partial \psi / \partial \sigma}{\partial \psi / \partial \mu} > -\frac{\partial T / \partial \sigma}{\partial T / \partial \mu} \implies \mu'_{\psi}(\sigma) > \mu'_{T}(\sigma)$$

This means that the first derivative of the Implicit Function $\mu_T(\sigma)$ determined by the definition of $Target = T(\sigma, \mu)$ is lower than the first derivative of the Implicit Function $\mu_{\psi}(\sigma)$ defined by the *Expected Utility Function* $= \psi(\sigma\ , \mu)$.

From the second Differential Condition we have:

$$DC2 = \frac{\partial R}{\partial \sigma}\frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \sigma}\frac{\partial R}{\partial \mu} > 0 \implies -\frac{\partial \psi / \partial \sigma}{\partial \psi / \partial \mu} < -\frac{\partial R / \partial \sigma}{\partial R / \partial \mu} \implies \mu'_{R}(\sigma) > \mu'_{\psi}(\sigma)$$

and by the third Differential Condition:

$$DC3 = \frac{\partial R}{\partial \sigma}\frac{\partial T}{\partial \mu} - \frac{\partial T}{\partial \sigma}\frac{\partial R}{\partial \mu} > 0 \implies -\frac{\partial \mu_T / \partial \sigma}{\partial \mu_T / \partial \mu} < -\frac{\partial R / \partial \sigma}{\partial R / \partial \mu} \implies \mu'_{R}(\sigma) > \mu'_{T}(\sigma)$$

Summing up:

$$(3.10) \qquad \mu'_T(\sigma) < \mu'_\psi(\sigma) < \mu'_R(\sigma)$$

which is an inequality between first derivatives of the Implicit Functions, which come from $T(\sigma,\mu), \psi(\sigma,\mu), R(\sigma,\mu)$ respectively. This inequality indicates the constraints that the curvature with respect to σ of these three Implicit Functions measured in a plane parallel to the plane (σ,μ) must satisfy.

Until now $R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)$, are supposed to be generic functions. It is interesting to discuss three cases of definition of *Risk* when the *return* is a Truncated Normal variable r_{TN} and we assume the *CRRA Expected Utility Function* . *Target* is defined, as usual, as *Expected value of* r_{TN}, or more briefly *Expected Return*.

4. CRRA Utility Function and the Truncated Normal Case

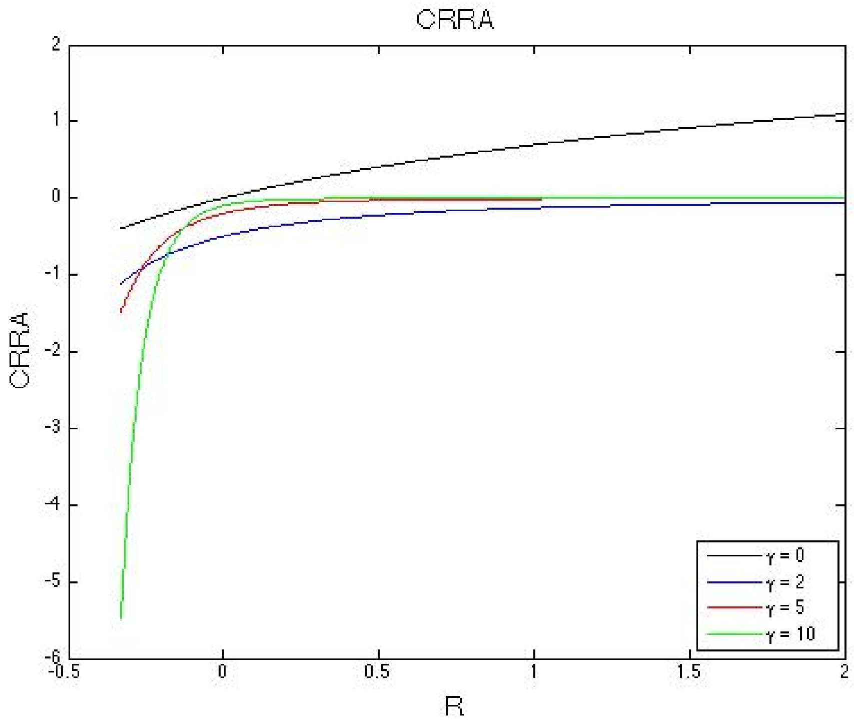
Consider a generic *CRRA* Utility Function:

$$(4.1) \qquad CRRA(\gamma) = \begin{cases} -W^{-\gamma}/\gamma, & \gamma > -1,\ \gamma \neq 0 \\ logW & \gamma = 0 \end{cases}$$

where W is defined by (2.1) and γ is a parameter that expresses an investor's sensitivity to risk.

The following Figure 4.1 shows the behavior of the CRRA with respect to different values of γ parameter.

Figure 4.1: Constant Relative Risk Aversion Utility Functions (CRRA)



$\gamma < -1$: *the investor is risk lover rather than risk-averse.*

$\gamma = -1$ *: means that the degree of risk aversion is zero, and the investor is indifferent between a risk-free choice and a risky choice so long as the arithmetic average expected return is the same.*

$\gamma = 0$ *: the investor is indifferent between a risk-free choice and a risky choice so long as the geometric average expected return is the same.*

$\gamma > 0$ *: the investor is risk-averse and calls a premium against his choice of a risky asset, the larger is the value of* γ *the greater the risk penalty.*

In this paper, we consider $\gamma = 2$. Without any loss of generality, we state $W_0 = 1$ in (4.1).

The value $r = -1$ represents a singular point for the (5.1), when $\gamma > 0$; this means that $r > -1$ is a condition that we have to pose. Furthermore, for $r < -1$ the *CRRA Utility Function* is not *risk-averse*.

Therefore, as particular case of $r \sim G(\sigma, \mu)$, where G is any two-parameter distribution, consider the return *r* as a Truncated Normal variable, that is *r* is constrained to assume values only in the interval $K = (k_1, k_2)$, with $-1 < k_1 < 0 < k_2 \leq \infty$ and $k_1 < \mu < k_2$; we call r_{TN} this constrained variable, where the suffix "*TN*" means Truncated Normal. In this paper the computations are done for $k_1 = -0.99,\ \ k_2 = \infty$. To define the density of the random variable r_{TN}, we use the following notations:

$$\phi(\xi) = \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}, \qquad \Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{\tau^2}{2}} d\tau$$

$$h_2 = \frac{k_2 - \mu}{\sigma}, \qquad h_1 = \frac{k_1 - \mu}{\sigma}, \qquad \Delta\Phi_K = \Phi(h_2) - \Phi(h_1)$$

Then, the density of the random variable r_{TN} is given by:

$$f(r_{TN}) = \begin{cases} \dfrac{\phi\left(\dfrac{r_{TN} - \mu}{\sigma}\right)}{\sigma \Delta\Phi_K} = \dfrac{e^{-(r_{TN}-\mu)^2 / 2\sigma^2}}{\int_{k_1}^{k_2} e^{-(x-\mu)^2 / 2\sigma^2} dx} & r_{TN} \in K \\ 0 & r_{TN} \notin K \end{cases}$$

and the *Expected Utility Function*, defined as $\psi(\sigma, \mu)$ is:

$$\psi(\sigma, \mu) \equiv E[CRRA(\gamma)] = -\frac{1}{\gamma} E\left[\frac{1}{(1 + r_{TN})^{\gamma}}\right] = -\frac{1}{\gamma\sigma\sqrt{2\pi}\Delta\Phi_K} \int_{k_1}^{k_2} \frac{e^{-(x-\mu)^2 / 2\sigma^2}}{(1 + x)^{\gamma}} dx$$

With the substitution $\tau = (x - \mu)/\sigma$ the function $\psi(\sigma, \mu)$ becomes:

(4.2) $$\psi(\sigma,\mu) = -\frac{1}{\gamma\sqrt{2\pi}\Delta\Phi_K}\int_{h_1}^{h_2}\frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^{\gamma}}d\tau = -\frac{1}{\gamma}\frac{\int_{h_1}^{h_2}\frac{e^{-\tau^2/2}}{(1+\mu+\sigma\tau)^{\gamma}}d\tau}{\int_{h_1}^{h_2}e^{-\tau^2/2}\,d\tau}$$

The case in Figure 3.1c is related to this CRRA Utility Function when *Risk* and *Target* are given by the transformation $R(\sigma,\mu) = \sigma,\ T(\sigma,\mu) = \mu$. This choice is not obviously the correct one.

Case 1: $R(\sigma,\mu) = Standard\ Deviation = SD_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

We have the transformation:

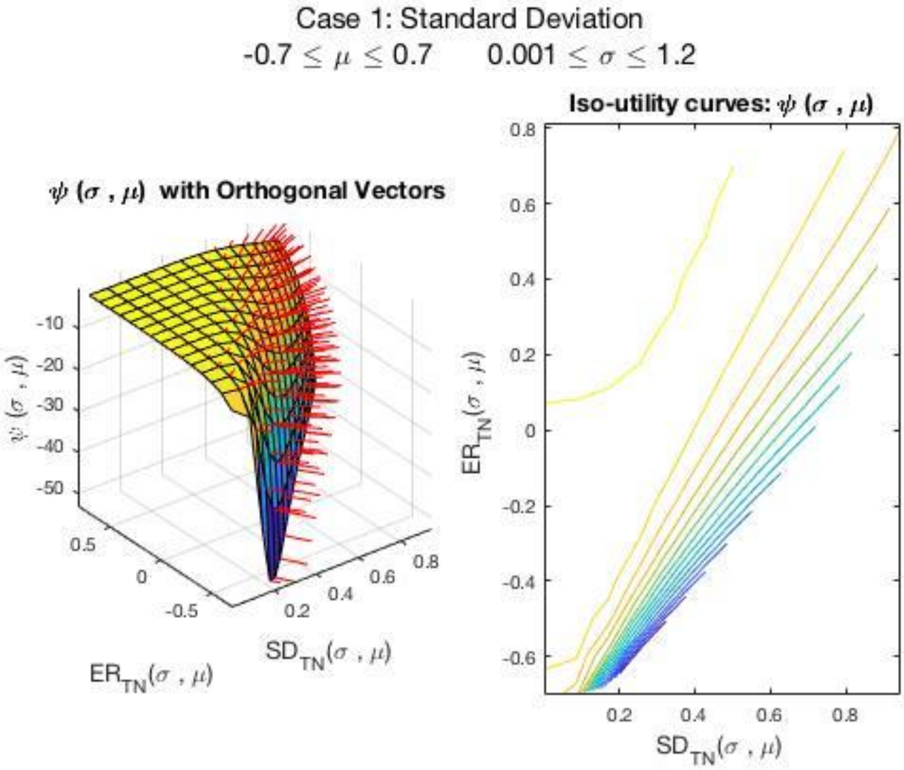
$$SD_{TN}(\sigma,\mu) = \sqrt{\frac{\int_{k_1}^{k_2}x^2e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}\,dx} - \left[\frac{\int_{k_1}^{k_2}xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}\,dx}\right]^2}$$

(4.3)

$$ER_{TN}\ (\sigma,\mu) = \frac{\int_{k_1}^{k_2}xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2}e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

with the parametric representation for $\psi(\sigma,\mu)$ and Iso-utility curves given by the following:

Figure 4.1: 3D and 2D



The Figure (4.1) shows that (4.3) definitions of *Risk* and *Target* respect the Differential Conditions (3.6). The Differential Conditions are greater than zero in the entire domain as it is shown in the Appendix C.

Case 2: $R(\sigma,\mu) = Value\ at\ Risk = VaR_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

$\alpha \quad = Confidence\ Level = 0.95$

In the Appendix D we compute the Value at Risk for a Truncated Normal, VaR_{TN}. We have the transformation:

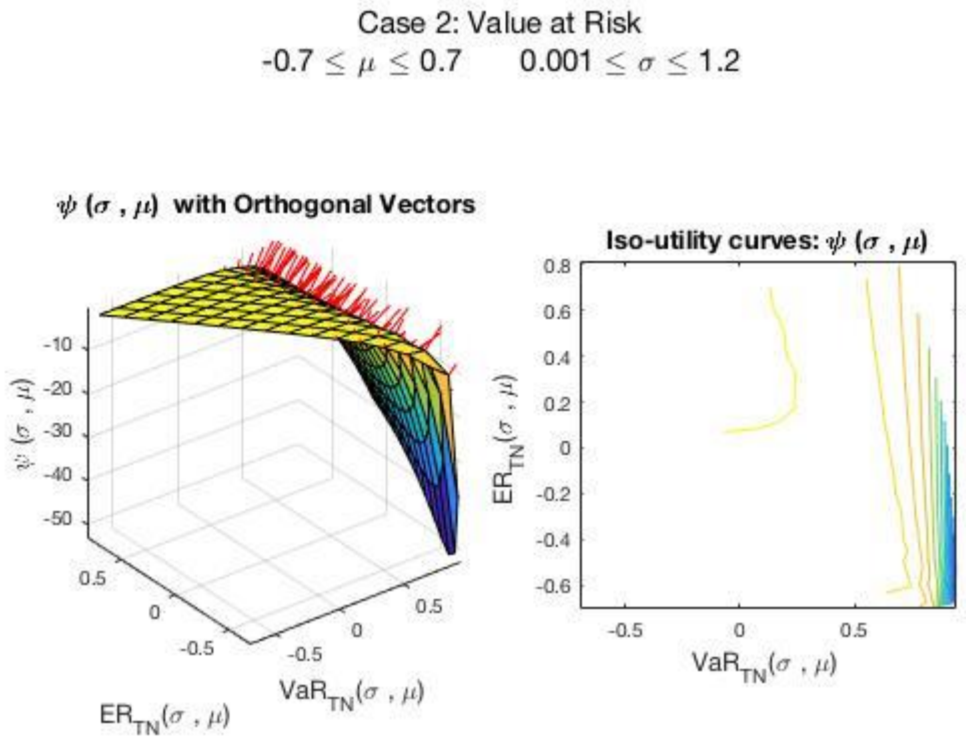
$$VaR_{TN}(\sigma,\mu) = -\mu - \sigma\Phi_{inv}\big(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\big)$$

(4.4)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

with the following parametric representation and Iso-utility curves for $\psi(\sigma,\mu)$:

Figure 4.2: 3D and 2D



In the VaR_{TN} case some Iso-utility curves have a negative slope: the Differential Conditions computed for VaR_{TN} are not respected in all the *3D* space $[VaR_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$. To be more precise, Differential Condition 2, relative to the component of the axis of ER_{TN} of the Normal unit vector in (3.5), is negative (see Appendix D). Writing ER_{TN} instead of T in (3.8) we have:

$$\frac{\partial\psi}{\partial ER_{TN}} < 0$$

that disagree with (3.8) constraint.

Case 3: $R(\sigma,\mu) = Expected\ Shortfall = ES_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

$\alpha \quad = Confidence\ Level = 0.95$

In Appendix E we compute the Expected Shortfall for a Truncated Normal, ES_{TN}. We have the transformation:

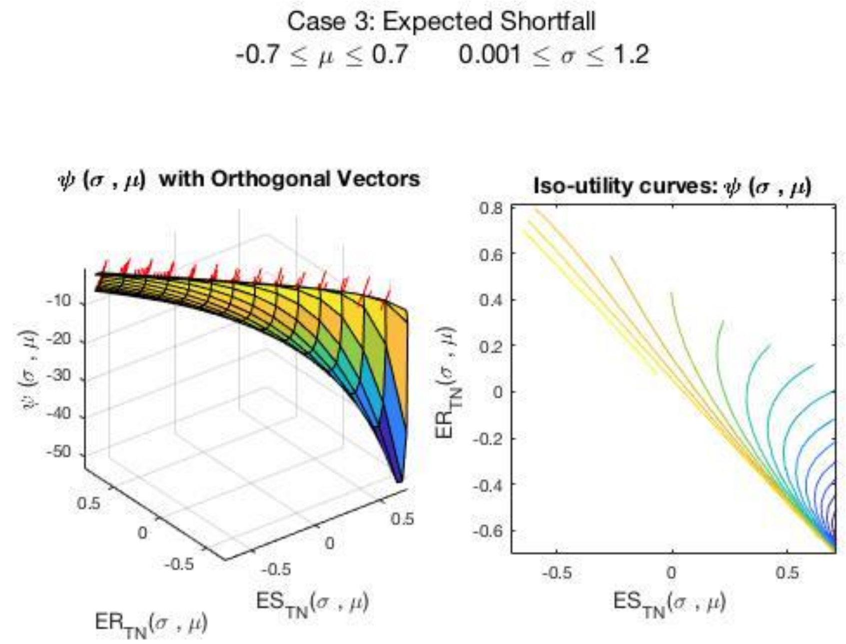
$$ES_{TN}(\sigma,\mu) = -\mu - \frac{\sigma\big[\phi(h_1) - \phi[\Phi_{inv}(b)]\big]}{(1-\alpha)\Delta\Phi_K}$$

(4.5)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

with the parametric representation and iso-utility curves for $\psi(\sigma,\mu)$:

Figure 4.3: 3D and 2D



that also demonstrates ES_{TN} does not respect the (3.6).

Indeed, the Differential Conditions computed for ES_{TN} are not respected in all the domain *3D* $[ES_{TN}(\sigma,\mu), ER_{TN}(\sigma,\mu), \psi(\sigma,\mu)]$. To be more precise, Differential Condition 2, relative to the component of the axis of ES_{TN} of the Normal unit vector in (3.5), is negative (see Appendix E).

The Quadratic Utility Function case is developed in Appendix F, G, H. This is an interesting case because it is possible to compute analytically the region in which the Differential Conditions are satisfied.

5. Conclusions

Starting with a *risk-averse Utility Function U(W*) with a wealth $W = W_0(1+r)$, where $r \sim G(\sigma,\mu)$ with G a generic distribution depending on two parameters, we consider the generic definitions of $Risk = R(\sigma,\mu), Target = T(\sigma,\mu)$. We find that the three functions $R(\sigma,\mu), T(\sigma,\mu)$ and *Expected Utility Function* $\psi(\sigma,\mu)$ must satisfy the Differential Conditions (3.6) so that $\psi(\sigma,\mu)$ has $\partial\psi/\partial R < 0, \partial\psi/\partial T > 0$, and the components of the orthogonal unit vectors are positive for $\psi - axis$ in the three dimension space $[R(\sigma,\mu), T(\sigma,\mu), \psi(\sigma,\mu)]$.

We present some cases in which the Differential Conditions show that not all the *Risk* and *Target* definitions imply iso-utility curves with positive slope, as we would be expected with the *risk – averse Utility Function*.

More precisely, if we consider the Truncated Normal case and define the *Target* as the Expected Return, ER_{TN}, then neither VaR nor Expected Shortfall (Case 2 and 3 of the previous section) respect the Differential Conditions: some Iso-utility curves have negative slope when the *CRRA Utility Function* is considered. Only the most elementary definition of *Risk*, the Standard Deviation (Case 1 of the previous section) respects the Differential Conditions.

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Appendix A. Proof of Theorem 2.1

Theorem 2.1*: Let $\succcurlyeq$ be an expected utility preference relation on all normal distributions $N(\mu, \sigma^2)$ for the return r. Then there exists a mean-variance Expected Utility Function $\psi(\sigma, \mu)$ which describes $\succcurlyeq$.*

In the case of risk-aversion, $\psi(\sigma, \mu)$ has the following partial derivatives and the first derivative of the implicit function $\mu_\psi(\sigma)$:

$$(A.1) \qquad \frac{\partial\psi(\sigma,\mu)}{\partial\mu} > 0, \quad \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} < 0, \qquad \Rightarrow \frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\frac{\partial\psi(\sigma,\mu)}{\partial\mu}} > 0$$

Proof:

Consider (2.1) here reported:

$$W = W_0(1 + r)$$

and without loss of generality pose $W_0 = 1$. We have:

$$r \backsim N(\mu, \sigma^2) \implies W \backsim N(1 + \mu, \sigma^2)$$

We prove at first the existence of $\psi(\sigma, \mu)$:

$$E[U(W)] = \int_{-\infty}^{\infty} \frac{U(W)e^{-(W-1-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dW$$

changing variable $z = (W - 1 - \mu)/\sigma$:

$$E[U(W)] = \int_{-\infty}^{\infty} \frac{U(1 + \mu + \sigma z)e^{-z^2/2}}{\sqrt{2\pi}} dz = \int_{-\infty}^{\infty} U(1 + \mu + \sigma z)\phi(z)\, dz = \psi(\sigma, \mu)$$

where $\phi(z)$ is the probability density function of the standard normal distribution.

Therefore, $E[U(W)]$ can be expressed as $\psi(\sigma, \mu)$, function of (σ, μ).

Now we can prove (A.1) when $U(W)$is *risk-averse*:

$$\frac{\partial\psi(\sigma,\mu)}{\partial\mu} = \int_{-\infty}^{\infty} U'\ (1+\mu+\sigma z)\phi(z)\,dz > 0$$

from (2.2). And:

$$\frac{\partial\psi(\sigma,\mu)}{\partial\sigma} = \int_{-\infty}^{\infty} zU'\ (1+\mu+\sigma z)\phi(z)\,dz$$

$$= \int_{-\infty}^{0} zU'\ (1+\mu+\sigma z)\phi(z)\,dz + \int_{0}^{\infty} zU'\ (1+\mu+\sigma z)\phi(z)\,dz$$
$$= \int_{0}^{\infty} z[U'\ (1+\mu+\sigma z)\ - U'\ (1+\mu-\sigma z)]\phi(z)\,dz$$

where the last line follows by the symmetry of $\phi(z)$.

By risk aversion $U''(W) < 0$ for all W, so that we have $U'\ (1+\mu+\sigma z) < U'\ (1+\mu-\sigma z)$ for $z > 0$, thus

$$\frac{\partial\psi(\sigma,\mu)}{\partial\sigma} < 0$$

i.e., risk aversion implies that investor likes higher expected returns and dislikes higher standard deviation. Differentiating implicitly:

$$\frac{d\mu_\psi(\sigma)}{d\sigma} = -\frac{\frac{\partial\psi(\sigma,\mu)}{\partial\sigma}}{\frac{\partial\psi(\sigma,\mu)}{\partial\mu}} > 0$$

Not surprisingly, indifference curves are upward in (σ,μ) Cartesian plane.

Appendix B. Some Mathematical notation.

We give the following definitions that will be useful in the next Appendices:

(B.1)

$$\tau = \frac{x-\mu}{\sigma}, \qquad h_2 = \frac{k_2-\mu}{\sigma}, \qquad h_1 = \frac{k_1-\mu}{\sigma},$$

$$I1 = \sigma\int_{h_1}^{h_2} e^{-\tau^2/2}\,d\tau, \;\; I2 = \int_{h_1}^{h_2} \tau e^{-\tau^2/2}\,d\tau, \quad I3 = \int_{h_1}^{h_2} \tau^2 e^{-\tau^2/2}\,d\tau$$

$$I4 = \sigma\int_{h_1}^{h_2} (\mu+\sigma\tau)e^{-\tau^2/2}\,d\tau, \;\; I5 = \int_{h_1}^{h_2} (\mu+\sigma\tau)\tau e^{-\tau^2/2}\,d\tau, \;\; I6 = \int_{h_1}^{h_2} (\mu+\sigma\tau)\tau^2 e^{-\tau^2/2}\,d\tau,$$

$$I7 = \sigma\int_{h_1}^{h_2} (\mu+\sigma\tau)^2 e^{-\tau^2/2}\,d\tau, \;\; I8 = \int_{h_1}^{h_2} (\mu+\sigma\tau)^2\tau e^{-\tau^2/2}\,d\tau, \;\; I9 = \int_{h_1}^{h_2} (\mu+\sigma\tau)^2\tau^2 e^{-\tau^2/2}\,d\tau.$$

Appendix C

Case 1: $R(\sigma,\mu) = Standard\ Deviation = SD_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

To compute the Standard Deviation and the Expected Return of the Truncated Normal variable, it is preferable to start with the following definitions:

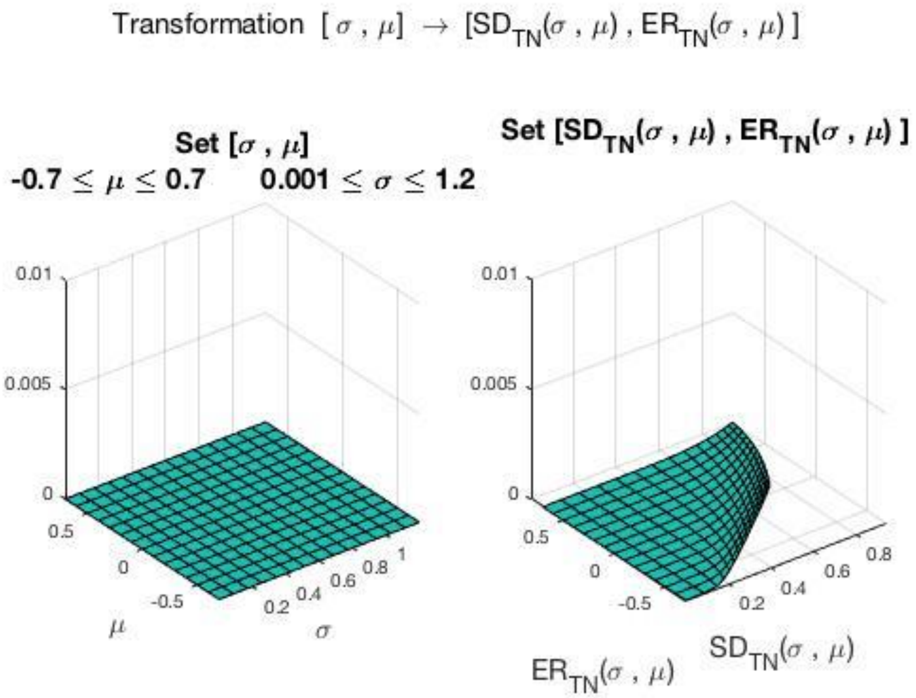
$$SD_{TN}(\sigma,\mu) = \sqrt{\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx} - \left[\frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right]^2}$$

(C.1)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}$$

The (C.1) formulas transform the set $[\sigma,\mu]$ into the set $[SD_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$ as it is possible to see from the following *Figure C.1*:

Figure C.1: Transformation $[\boldsymbol{\sigma}, \boldsymbol{\mu}] \rightarrow [\mathbf{SD_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu}),\ \mathbf{ER_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu})]$



The partial derivatives, using (B.1) are:

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\sigma} = \frac{\partial}{\partial\sigma}\left\{\frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right\}$$

$$= \frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} x\,\frac{(x-\mu)^2}{\sigma^3}\,e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$-\frac{\left[\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} \frac{(x-\mu)^2}{\sigma^3}\,e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\sigma} = \frac{I1 * I6 - I3 * I4}{(I1)^2};$$

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\mu} = \frac{\partial}{\partial\mu}\left\{\frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx}\right\}$$

$$= \frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} x\left(\frac{x-\mu}{\sigma^2}\right)e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$-\frac{\left[\int_{k_1}^{k_2} xe^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} \left(\frac{x-\mu}{\sigma^2}\right)e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$\frac{\partial ER_{TN}(\sigma,\mu)}{\partial\mu} = \frac{I1 * I5 - I2 * I4}{(I1)^2};$$

To compute the partial derivatives of SD_{TN} we consider:

$$\frac{\partial}{\partial\sigma}\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx} =$$

$$= \frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} x^2 \frac{(x-\mu)^2}{\sigma^3} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$- \frac{\left[\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} \frac{(x-\mu)^2}{\sigma^3} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$= \frac{I1 * I9 - I3 * I7}{(I1)^2} \Longrightarrow:$$

$$\frac{\partial SD_{TN}(\sigma,\mu)}{\partial\sigma} = \frac{1}{2\,SD_{TN}(\sigma,\mu)}\left[\frac{I1 * I9 - I3 * I7}{(I1)^2} - 2\,ME_{TN}(\sigma,\mu)\frac{\partial ME_{TN}(\sigma,\mu)}{\partial\sigma}\right]$$

and:

$$\frac{\partial}{\partial\mu}\frac{\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx} =$$

$$= \frac{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2} x^2 \left(\frac{x-\mu}{\sigma^2}\right) e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$-\frac{\left[\int_{k_1}^{k_2} x^2 e^{-(x-\mu)^2/2\sigma^2}\,dx\right]\left[\int_{k_1}^{k_2}\left(\frac{x-\mu}{\sigma^2}\right) e^{-(x-\mu)^2/2\sigma^2}\,dx\right]}{\left[\int_{k_1}^{k_2} e^{-(x-\mu)^2/2\sigma^2}\,dx\right]^2}$$

$$=\frac{I1 * I8 - I2 * I7}{(I1)^2} \Longrightarrow$$

$$\frac{\partial SD_{TN}(\sigma,\mu)}{\partial\mu}=\frac{1}{2\,SD_{TN}(\sigma,\mu)}\left[\frac{I1I8-I2I7}{I1^2}-2\,ME_{TN}(\sigma,\mu)\frac{\partial ME_{TN}(\sigma,\mu)}{\partial\mu}\right]$$

It is possible now to compute and to graph the Differential Conditions (3.6).
The following three Figures show us that (3.6) are satisfied, all Differential Conditions are greater than zero.

Figure C.2: Differential Condition 1 for $[\mathbf{SD_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu}),\ \mathbf{ER_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu})]$

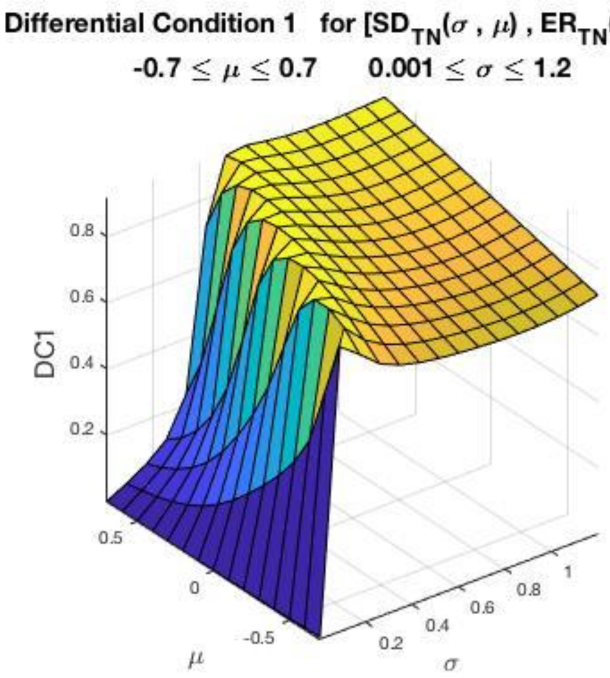


Figure C.3: Differential Condition 2 for $[\mathbf{SD_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu}),\ \mathbf{ER_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu})]$

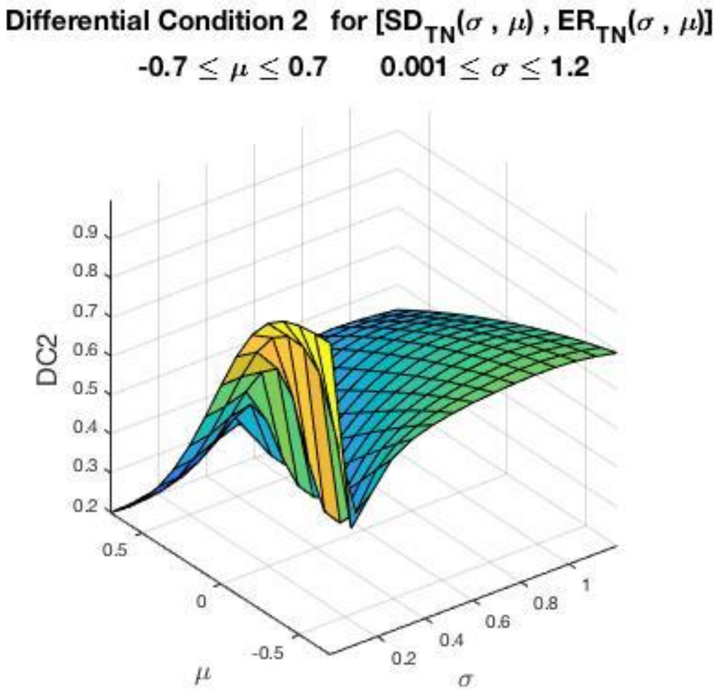
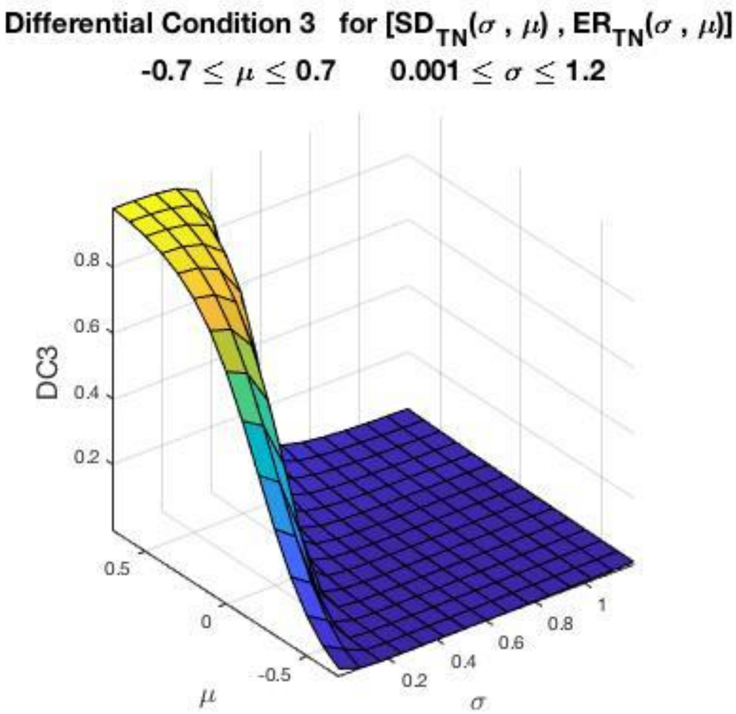


Figure C.4: Differential Condition 3 for $[\mathbf{SD_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu}),\ \mathbf{ER_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu})]$



Appendix D

Case 2: $R(\sigma,\mu) = Value\ at\ Risk = \ VaR_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \ \gamma\ = 2.$

$\alpha \qquad = Confidence\ Level = 0.95$

It is possible to analyze the behavior of $VaR_{TN} \equiv VaR_{TN}(\sigma,\mu)$. Starting from its definitions:

$$1-\alpha = \frac{1}{\sigma\sqrt{2\pi}\Delta\Phi_K}\int_{k_1}^{-VaR_{TN}} e^{-(\xi-\mu)^2/_{2\sigma^2}}\,d\xi$$

and using (B1) we have:

$$1-\alpha = \frac{1}{\Delta\Phi_K}\left[\Phi\left(\frac{-VaR_{TN}-\mu}{\sigma}\right) - \Phi(h_1)\right] \Rightarrow \Phi\left(\frac{-VaR_{TN}-\mu}{\sigma}\right) = (1-\alpha)\Delta\Phi_K + \Phi(h_1);$$

$$\Phi\left(\frac{-VaR_{TN}-\mu}{\sigma}\right) = (1-\alpha)\Phi(h_2) - (1-\alpha)\Phi(h_1) + \Phi(h_1) = \alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)$$

$$VaR_{TN}(\sigma,\mu) = -\mu - \sigma\Phi_{inv}\big(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\big)$$

obtaining the transformation:

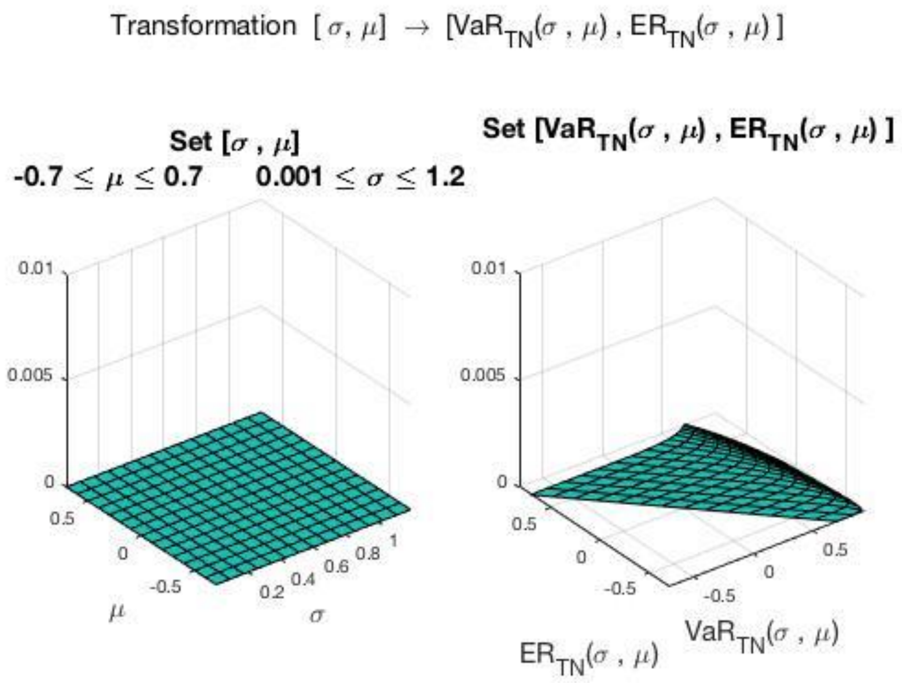
$$VaR_{TN}(\sigma,\mu) = -\mu - \sigma\Phi_{inv}\big(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\big)$$

(D.1)

$$ER_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} xe^{-(x-\mu)^2/_{2\sigma^2}}\,dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2/_{2\sigma^2}}\,dx}$$

The (D.1) transforms the set $[\sigma,\mu]$ in the set $[VaR_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$ as it is possible to see in the following *Figure D.1*:

Figure D.1: Transformation $[\boldsymbol{\sigma}, \boldsymbol{\mu}] \rightarrow [\mathbf{VaR_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu}),\ \mathbf{ER_{TN}}(\boldsymbol{\sigma}, \boldsymbol{\mu})]$



Defining:

$$(D.2) \qquad b = \alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\,, \qquad c = \Phi_{inv}(b)\,, \quad \Phi_{inv}(b) = -(VaR_{TN} + \mu)/\sigma$$

and computing:

$$(D.3) \qquad \frac{\partial \mathrm{b}}{\partial \mu} = -\frac{\alpha}{\sigma}\phi(h_1) - \frac{(1-\alpha)}{\sigma}\phi(h_2); \quad \frac{\partial \mathrm{b}}{\partial \sigma} = -\frac{\alpha h_1}{\sigma}\phi(h_1) - \frac{(1-\alpha)h_2}{\sigma}\phi(h_2)$$

we can use the Theorem of derivative of the inverse function:

$$\frac{d\Phi_{inv}(b)}{db} = \frac{1}{\frac{d\Phi(c)}{dc}} \quad iif \quad \frac{d\Phi(c)}{dc} \neq 0$$

to compute the partial derivatives of:

$$(D.4) \qquad \frac{\partial \Phi_{inv}(b)}{\partial \sigma} = \frac{d\Phi_{inv}(b)}{db} \cdot \frac{\partial \mathrm{b}}{\partial \sigma} = \frac{1}{\frac{d\Phi(c)}{dc}} \cdot \frac{\partial \mathrm{b}}{\partial \sigma}$$

By the definition of Φ:

$$\frac{d\Phi(c)}{dc} = \frac{d}{dc}\int_{-\infty}^{c} \phi(\tau)\, d\tau = \phi(c) = \phi\big(\Phi_{inv}(b)\big)$$

we have:

$$\frac{\partial \Phi_{inv}(b)}{\partial \sigma} = \frac{1}{\phi\big(\Phi_{inv}(b)\big)} \cdot \frac{\partial \mathrm{b}}{\partial \sigma}$$

and consequently:

$$\frac{\partial \Phi_{inv}(b)}{\partial \mu} = \frac{1}{\phi\big(\Phi_{inv}(b)\big)} \cdot \frac{\partial \mathrm{b}}{\partial \mu}$$

So we can compute the partial derivatives of VaR_{TN}:

$$\frac{\partial VaR_{TN}}{\partial \mu} = -1 - \sigma \frac{\left(-\frac{\alpha}{\sigma}\phi(h_1) - \frac{(1-\alpha)}{\sigma}\phi(h_2)\right)}{\phi\left(\Phi_{inv}(b)\right)}$$

$$= -1 + \frac{\alpha\phi(h_1) + (1-\alpha)\phi(h_2)}{\phi(-(VaR_{TN} + \mu)/\sigma)}$$

$$\frac{\partial VaR_{TN}}{\partial \sigma} = -\Phi_{inv}\left(\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2)\right) - \sigma \frac{\left(-\frac{\alpha h_1}{\sigma}\phi(h_1) - \frac{(1-\alpha)h_2}{\sigma}\phi(h_2)\right)}{\phi\left(\Phi_{inv}(b)\right)}$$

$$= \frac{VaR_{TN} + \mu}{\sigma} - \sigma \frac{\left(-\frac{\alpha h_1}{\sigma}\phi(h_1) - \frac{(1-\alpha)h_2}{\sigma}\phi(h_2)\right)}{\phi\left(\Phi_{inv}(b)\right)}$$

$$= \frac{VaR_{TN} + \mu}{\sigma} + \frac{\left(\alpha h_1\phi(h_1) + (1-\alpha)h_2\phi(h_2)\right)}{\phi(-(VaR_{TN} + \mu)/\sigma)}$$

Now, it is possible to compute the Differential Conditions (3.6) and to graph them. DC1 is satisfied, it is > 0.

Figure D.2: Differential Condition 1 for $[\mathbf{VaR_{TN}}(\boldsymbol{\sigma},\boldsymbol{\mu}),\ \mathbf{ER_{TN}}(\boldsymbol{\sigma},\boldsymbol{\mu})]$

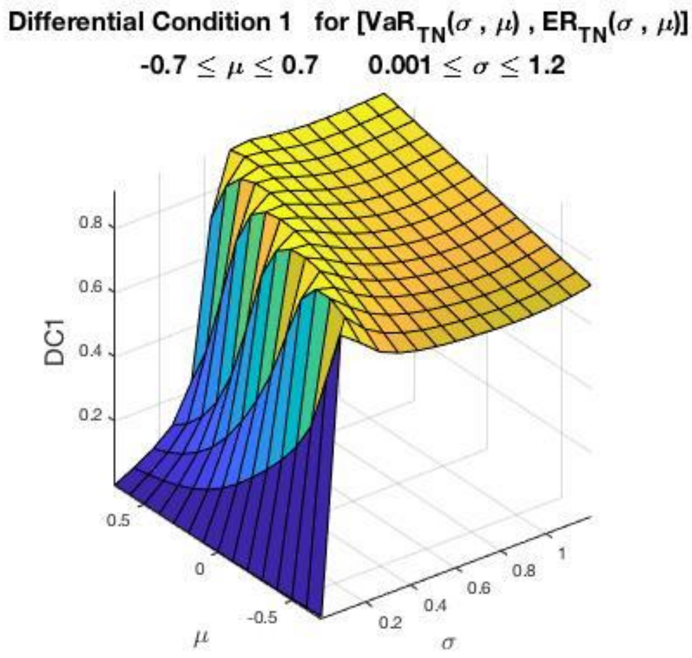


Figure D.3: Differential Condition 2 for $[\mathbf{VaR_{TN}(\sigma, \mu)},\ \mathbf{ER_{TN}(\sigma, \mu)}]$

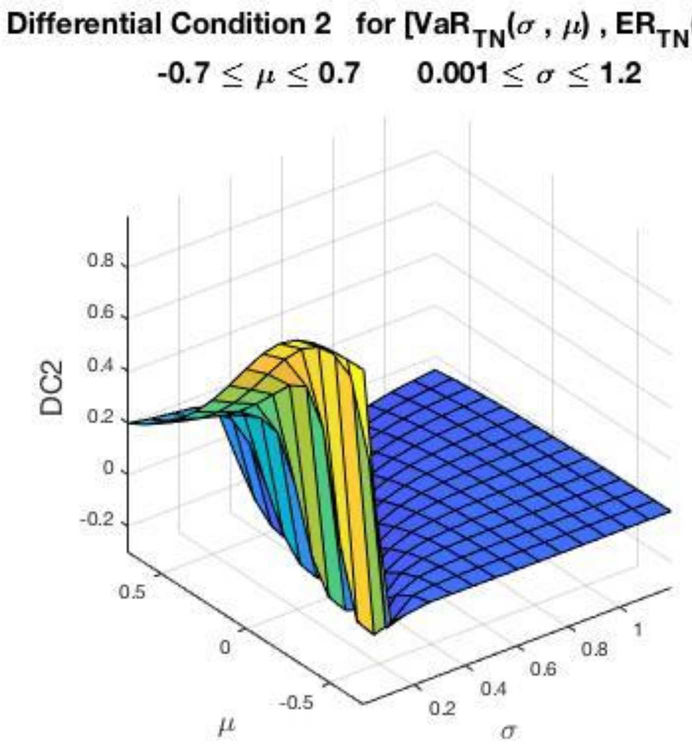


Table D.1: Values of the Differential Condition 2 for $[\mathbf{VaR_{TN}}(\boldsymbol{\sigma},\boldsymbol{\mu}),\ \mathbf{ER_{TN}}(\boldsymbol{\sigma},\boldsymbol{\mu})]$

0,700	0,199	0,184	0,172	0,162	0,133	-0,110	-0,298	-0,270	-0,209	-0,155	-0,116	-0,089	-0,071	-0,059	-0,050
0,600	0,237	0,218	0,204	0,190	0,127	-0,215	-0,295	-0,240	-0,176	-0,126	-0,093	-0,071	-0,057	-0,048	-0,041
0,500	0,284	0,261	0,243	0,223	0,076	-0,277	-0,275	-0,206	-0,144	-0,101	-0,073	-0,056	-0,045	-0,038	-0,034
0,400	0,342	0,313	0,291	0,260	-0,049	-0,292	-0,244	-0,170	-0,114	-0,078	-0,057	-0,044	-0,036	-0,031	-0,028
0,300	0,414	0,379	0,350	0,291	-0,192	-0,279	-0,205	-0,134	-0,088	-0,060	-0,044	-0,035	-0,029	-0,025	-0,023
0,200	0,500	0,458	0,421	0,275	-0,264	-0,247	-0,164	-0,102	-0,065	-0,045	-0,034	-0,027	-0,023	-0,020	-0,019
0,100	0,600	0,552	0,501	0,119	-0,274	-0,203	-0,123	-0,074	-0,048	-0,034	-0,026	-0,021	-0,018	-0,016	-0,015
0,000	0,706	0,654	0,578	-0,117	-0,249	-0,155	-0,087	-0,051	-0,034	-0,025	-0,019	-0,016	-0,014	-0,013	-0,013
-0,100	0,808	0,754	0,616	-0,231	-0,201	-0,107	-0,057	-0,035	-0,024	-0,018	-0,014	-0,012	-0,011	-0,011	-0,010
-0,200	0,890	0,837	0,491	-0,241	-0,142	-0,067	-0,036	-0,023	-0,016	-0,013	-0,011	-0,010	-0,009	-0,009	-0,008
-0,300	0,946	0,887	0,080	-0,194	-0,086	-0,038	-0,021	-0,014	-0,011	-0,009	-0,008	-0,007	-0,007	-0,007	-0,007
-0,400	0,977	0,898	-0,168	-0,122	-0,043	-0,020	-0,012	-0,008	-0,007	-0,006	-0,006	-0,005	-0,005	-0,005	-0,006
-0,500	0,992	0,852	-0,176	-0,054	-0,017	-0,008	-0,006	-0,005	-0,004	-0,004	-0,004	-0,004	-0,004	-0,004	-0,005
-0,600	0,998	0,577	-0,085	-0,014	-0,004	-0,002	-0,002	-0,002	-0,002	-0,002	-0,003	-0,003	-0,003	-0,003	-0,004
-0,700	1,000	-0,034	-0,007	0,004	0,003	0,002	0,001	-0,000	-0,001	-0,001	-0,002	-0,002	-0,002	-0,003	-0,003
μ↑σ→	**0,001**	**0,087**	**0,172**	**0,258**	**0,344**	**0,429**	**0,515**	**0,601**	**0,686**	**0,772**	**0,857**	**0,943**	**1,029**	**1,114**	**1,200**

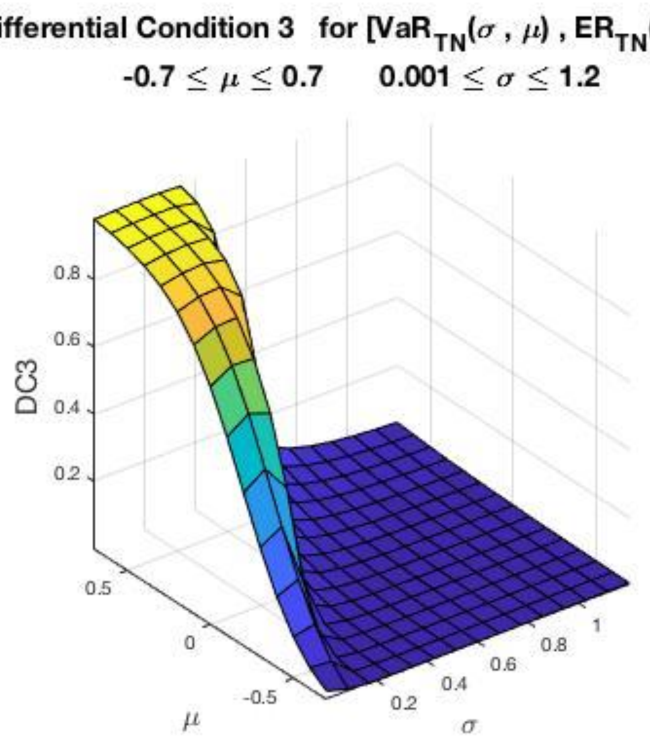
Condition DC2 is not satisfied, as it is possible to see from Figure D.3 and *Table D.1*, where its values are reported. This means that this transformation, even if it is based on *Risk Averse Utility Function*, does not preserve the concavity property and there are regions of its domain where

$$\frac{\partial \psi}{\partial ER_{TN}} < 0$$

(see (3.8) and pose ER_{TN} instead of T) .

DC3 is satisfied, it is > *0*.

Figure D.4: Differential Condition 3 for $[\mathbf{VaR_{TN}(\sigma,\mu)},\ \mathbf{ER_{TN}(\sigma,\mu)}]$



Appendix E

Case 3: $R(\sigma,\mu) = Expected\ Shortfall = ES_{TN}(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = ER_{TN}(\sigma,\mu)$

$\psi(\sigma,\mu) = Expected\ CRRA\ Utility\ Function\ with\ \gamma = 2.$

$\alpha \quad = Confidence\ Level = 0.95$

Starting from the definitions of Expected Shortfall of a Truncated Normal:

$$-ES_{TN} = \frac{1}{(1-\alpha)\Delta\Phi_K}\int_{k_1}^{-VaR_{TN}} \frac{xe^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,dx$$

we have:

$$-ES_{TN} = \frac{1}{(1-\alpha)\Delta\Phi_K}\int_{h_1}^{(-VaR_{TN}-\mu)/\sigma} (\sigma\tau+\mu)\phi(\tau)\,d\tau$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\int_{h_1}^{(-VaR_{TN}-\mu)/\sigma} \tau\phi(\tau)\,d\tau + \mu\int_{h_1}^{(-VaR_{TN}-\mu)/\sigma} \phi(\tau)\,d\tau\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma[-\phi(\tau)]_{h_1}^{(-VaR_{TN}-\mu)/\sigma} + \mu[\Phi(-(VaR_{TN}+\mu)/\sigma) - \Phi(h_1)]\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\{\sigma[\phi(h_1) - \phi(-(VaR_{TN}+\mu)/\sigma)] + \mu[\Phi(-(VaR_{TN}+\mu)/\sigma) - \Phi(h_1)]\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\{\sigma\big[\phi(h_1) - \phi[\Phi_{inv}(b)]\big] + \mu[\Phi[\Phi_{inv}(b)] - \Phi(h_1)]\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\{\sigma\big[\phi(h_1) - \phi[\Phi_{inv}(b)]\big] + \mu[\mathrm{b} - \Phi(h_1)]\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\{\sigma\big[\phi(h_1) - \phi[\Phi_{inv}(b)]\big] + \mu[\alpha\Phi(h_1) + (1-\alpha)\Phi(h_2) - \Phi(h_1)]\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right] + \mu(1-\alpha)[\Phi(h_2) - \Phi(h_1)]\right\}$$

$$= \frac{1}{(1-\alpha)\Delta\Phi_K}\left\{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right] + \mu(1-\alpha)\Delta\Phi_K\right\}$$

and finally:

$$ES_{TN} \ = \ -\mu - \frac{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right]}{(1-\alpha)\Delta\Phi_K}$$

Therefore, the transformations become:

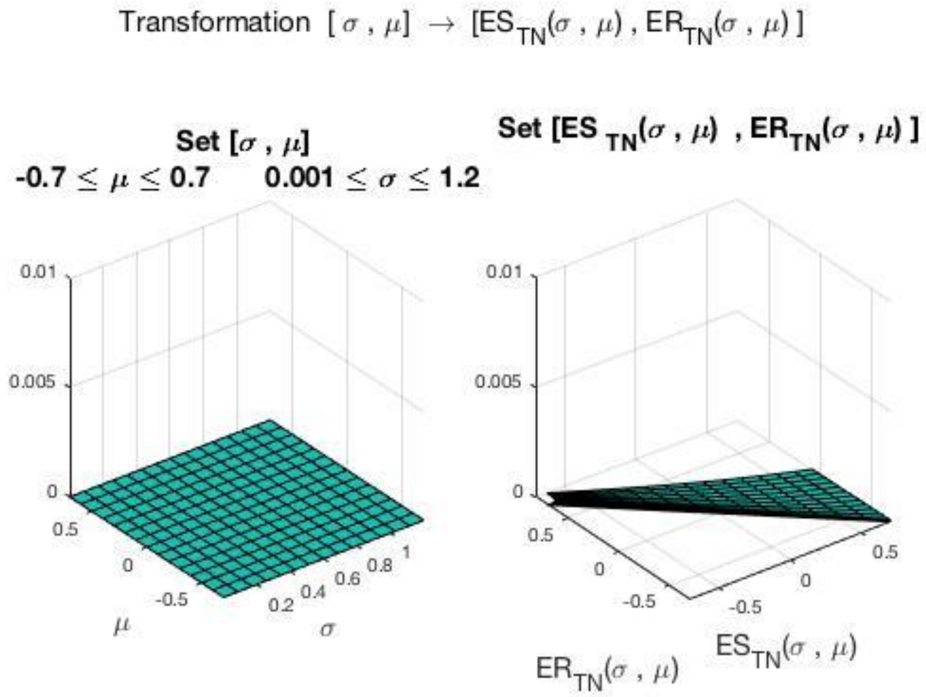
$$ES_{TN}(\sigma,\mu) \ = \ -\mu - \frac{\sigma\left[\phi(h_1) - \phi[\Phi_{inv}(b)]\right]}{(1-\alpha)\Delta\Phi_K}$$

(E.1)

$$ME_{TN}(\sigma,\mu) = \frac{\int_{k_1}^{k_2} x e^{-(x-\mu)^2 / 2\sigma^2}\, dx}{\int_{k_1}^{k_2} e^{-(x-\mu)^2 / 2\sigma^2}\, dx}$$

Formulas (E.1) transform the set $[\sigma,\mu]$ into the set $[ES_{TN}(\sigma,\mu),\ ER_{TN}(\sigma,\mu)]$ as it is possible to see from the following representations:

Figure E.1: Transformation $[\sigma, \mu] \rightarrow [ES_{TN}(\sigma, \mu),\ ER_{TN}(\sigma, \mu)]$



Using the definitions (D.2) and the Theorem of derivative of the inverse functions (D.4):

$$\frac{d\Phi_{inv}(b)}{db} = \frac{1}{\frac{d\Phi(c)}{dc}} \quad iif \quad \frac{d\Phi(c)}{dc} \neq 0$$

we can compute the partial derivatives of:

$$\frac{\partial \phi\big(\Phi_{inv}(b)\big)(\sigma,\mu)}{\partial\sigma} = \frac{\partial \phi\big(\Phi_{inv}(b)\big)}{\partial\sigma} = \frac{1}{\sqrt{2\pi}} \frac{\partial \exp\left(-\, \Phi_{inv}^{2}(b)/_{2}\right)}{\partial\sigma}$$

We have:

$$\frac{\partial \phi\big(\Phi_{inv}(b)\big)}{\partial\sigma} = -\frac{1}{\sqrt{2\pi}} exp\left(-\, \Phi_{inv}^{2}(b)/_{2}\right) \cdot \Phi_{inv}(b) \cdot \frac{1}{\frac{d\Phi(c)}{dc}} \cdot \frac{\partial \mathrm{b}}{\partial\sigma}$$

$$= -\phi\big(\Phi_{inv}(b)\big) \cdot \Phi_{inv}(b) \cdot \frac{1}{\frac{d\Phi(c)}{dc}} \cdot \frac{\partial \mathrm{b}}{\partial\sigma}$$

By the definition of Φ:

$$\frac{d\Phi(c)}{dc} = \frac{d}{dc}\int_{-\infty}^{c} \phi(\tau)\, d\tau = \phi(c) = \phi\big(\Phi_{inv}(b)\big)$$

we have:

$$\frac{\partial \phi\big(\Phi_{inv}(b)\big)}{\partial\sigma} = -\phi\big(\Phi_{inv}(b)\big) \cdot \Phi_{inv}(b) \cdot \frac{1}{\phi\big(\Phi_{inv}(b)\big)} \cdot \frac{\partial \mathrm{b}}{\partial\sigma} = -\Phi_{inv}(b) \cdot \frac{\partial \mathrm{b}}{\partial\sigma}$$

Using (D.3):

$$(E.2) \qquad \frac{\partial \phi\big(\Phi_{inv}(b)\big)}{\partial\sigma} = \frac{\Phi_{inv}(b)}{\sigma}\left[\alpha h_1 \phi(h_1) + (1-\alpha) h_2 \phi(h_2)\right]$$

and with the same rationale:

$$(E.3) \qquad \frac{\partial \phi\big(\Phi_{inv}(b)\big)}{\partial \mu} = \frac{\Phi_{inv}(b)}{\sigma}\left[\alpha \phi(h_1) + (1-\alpha)\phi(h_2)\right]$$

We rewrite ES_{TN} as:

$$ES_{TN} \; = \; -\mu - \frac{\sigma\big[\phi(h_1) - \phi[\Phi_{inv}(b)]\big]}{(1-\alpha)\Delta\Phi_K} = -\mu - \frac{\sigma\left[e^{-(k_1-\mu)^2/2\sigma^2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]}{(1-\alpha)\int_{h_1}^{h_2} e^{-\tau^2/2}\, d\tau}$$

that allows us to compute the partial derivatives of ES_{TN}:

$$\frac{\partial ES_{TN}}{\partial \mu} = -1 - \frac{\partial}{\partial \mu}\left\{\frac{\sigma^2\left[e^{-(k_1-\mu)^2/2\sigma^2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]}{(1-\alpha)I1}\right\}$$

$$= -1 - \frac{\sigma^2\left\{I1\left[\frac{h_1}{\sigma}e^{-h_1{}^2/2} - \sqrt{2\pi}\frac{\partial\phi\big(\Phi_{inv}(b)\big)}{\partial\mu}\right] - \left[e^{-h_1{}^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]\int_{h_1}^{h_2}\tau e^{-\tau^2/2}\, d\tau\right\}}{(1-\alpha)(I1)^2}$$

$$= -1 - \frac{\sigma^2\left\{I1\left[\frac{h_1}{\sigma}e^{-h_1{}^2/2} - \sqrt{2\pi}\frac{\partial\phi\big(\Phi_{inv}(b)\big)}{\partial\mu}\right] - \left[e^{-h_1{}^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right] I2\right\}}{(1-\alpha)(I1)^2}$$

Here, we can use (E.3) instead of $\partial\phi\big(\Phi_{inv}(b)\big)/\partial\mu$.

$$\frac{\partial ES_{TN}}{\partial \sigma} = -\frac{\partial}{\partial \sigma}\left\{\frac{\sigma^2\left[e^{-(k_1-\mu)^2/2\sigma^2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]}{(1-\alpha)I1}\right\}$$

$$= -\frac{I1\left\{2\sigma\left[e^{-h_1{}^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right] + \sigma^2\left[\frac{h_1^2}{\sigma}e^{-h_1{}^2/2} - \sqrt{2\pi}\frac{\partial\phi\left(\Phi_{inv}(b)\right)}{\partial\sigma}\right]\right\} + \cdots}{(1-\alpha)(I1)^2}$$

$$\frac{\ldots - \sigma^2\left[e^{-h_1{}^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right]\int_{h_1}^{h_2}\tau^2 e^{-\tau^2/2}\,d\tau}{(1-\alpha)(I1)^2}$$

$$= -\frac{\sigma\left\{\left[e^{-h_1{}^2/2} - e^{-[\Phi_{inv}(b)]^2/2\sigma^2}\right][2I1 - \sigma I3] + \sigma I1\left[\frac{h_1^2}{\sigma}e^{-h_1{}^2/2} - \sqrt{2\pi}\frac{\partial\phi\left(\Phi_{inv}(b)\right)}{\partial\sigma}\right]\right\}}{(1-\alpha)(I1)^2}$$

and then we have the following figures and tables.

DC1 is satisfied, it is > 0.

Figure E.2: Differential Condition 1 for $[\mathbf{ES_{TN}(\sigma, \mu)},\ \mathbf{ER_{TN}(\sigma, \mu)}]$

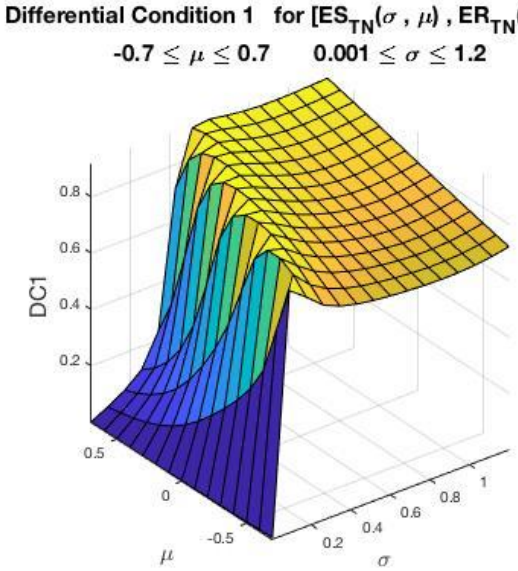


Figure E.3: Differential Condition 2 for $[ES_{TN}(\sigma, \mu),\ ER_{TN}(\sigma, \mu)]$

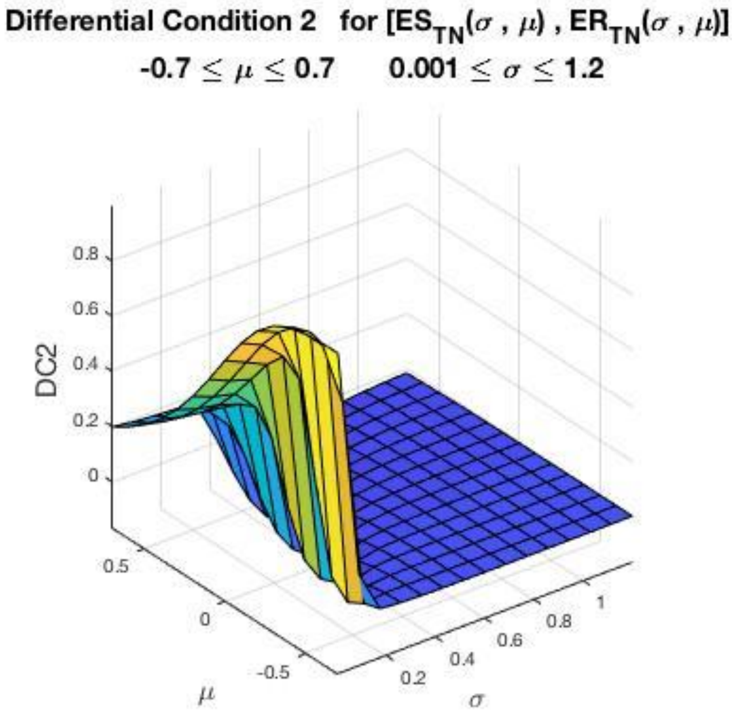
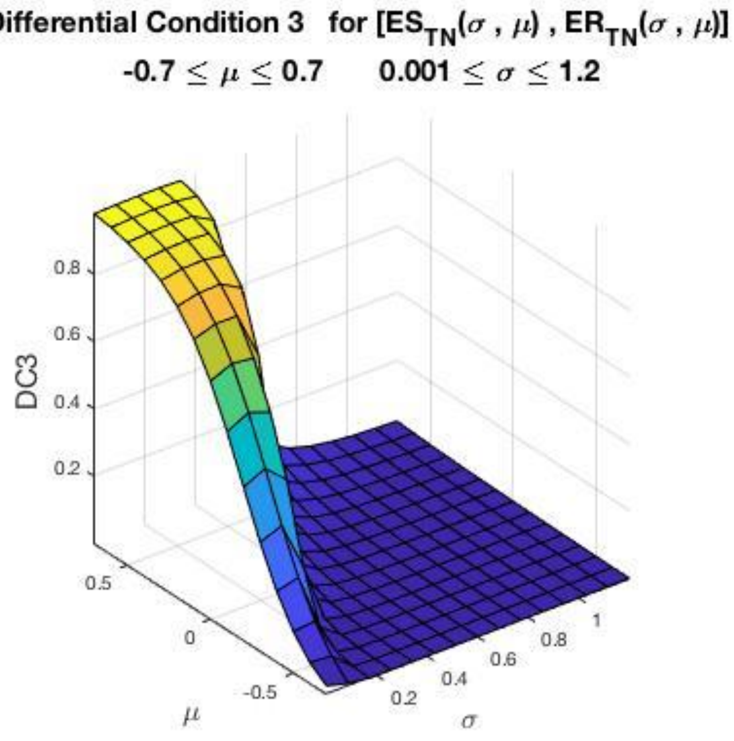


Table E.1: Values of the Differential Condition 2 for $[\mathbf{ES_{TN}}(\sigma, \mu),\ \mathbf{ER_{TN}}(\sigma, \mu)]$

0,700	0,199	0,188	0,181	0,176	0,163	0,005	-0,166	-0,145	-0,099	-0,065	-0,044	-0,031	-0,023	-0,018	-0,015
0,600	0,237	0,223	0,214	0,209	0,175	-0,074	-0,164	-0,121	-0,077	-0,049	-0,033	-0,023	-0,018	-0,014	-0,012
0,500	0,284	0,267	0,256	0,249	0,163	-0,135	-0,147	-0,096	-0,058	-0,036	-0,024	-0,017	-0,013	-0,011	-0,009
0,400	0,342	0,321	0,308	0,296	0,094	-0,155	-0,122	-0,072	-0,042	-0,026	-0,018	-0,013	-0,010	-0,008	-0,007
0,300	0,414	0,388	0,373	0,346	-0,027	-0,146	-0,093	-0,052	-0,030	-0,018	-0,013	-0,009	-0,007	-0,006	-0,006
0,200	0,500	0,470	0,451	0,374	-0,111	-0,121	-0,067	-0,035	-0,020	-0,013	-0,009	-0,007	-0,005	-0,005	-0,004
0,100	0,600	0,566	0,541	0,298	-0,133	-0,090	-0,044	-0,023	-0,013	-0,008	-0,006	-0,005	-0,004	-0,004	-0,003
0,000	0,707	0,670	0,633	0,083	-0,116	-0,059	-0,027	-0,014	-0,008	-0,005	-0,004	-0,003	-0,003	-0,003	-0,002
-0,100	0,808	0,772	0,701	-0,064	-0,083	-0,035	-0,015	-0,008	-0,005	-0,003	-0,002	-0,002	-0,002	-0,002	-0,002
-0,200	0,890	0,856	0,657	-0,096	-0,049	-0,017	-0,007	-0,004	-0,002	-0,002	-0,001	-0,001	-0,001	-0,001	-0,001
-0,300	0,946	0,911	0,321	-0,071	-0,022	-0,007	-0,003	-0,001	-0,001	-0,001	-0,001	-0,001	-0,001	-0,001	-0,001
-0,400	0,977	0,932	0,015	-0,032	-0,006	-0,001	0,000	0,000	0,000	0,000	-0,000	-0,000	-0,000	-0,000	-0,000
-0,500	0,992	0,913	-0,038	-0,005	0,002	0,002	0,002	0,001	0,001	0,001	0,000	0,000	0,000	-0,000	-0,000
-0,600	0,998	0,750	-0,004	0,006	0,005	0,003	0,002	0,002	0,001	0,001	0,001	0,001	0,000	0,000	0,000
-0,700	1,000	0,169	0,016	0,008	0,005	0,004	0,003	0,002	0,002	0,001	0,001	0,001	0,001	0,000	0,000
μ↑σ→	**0,001**	**0,087**	**0,172**	**0,258**	**0,344**	**0,429**	**0,515**	**0,601**	**0,686**	**0,772**	**0,857**	**0,943**	**1,029**	**1,114**	**1,200**

Also in this case condition DC2 is not satisfied, see *Figure E.3* and *Table E.1*, and we can conclude with the same considerations done for *Differential Condition 2* of Appendix D. DC3 is satisfied, it is > *0*.

Figure E.4: Differential Condition 3 for $[ES_{TN}(\sigma,\mu),\ ME_{TN}(\sigma,\mu)]$



Appendix F. Quadratic Utility Function

Consider the following general Quadratic Utility Function (QUF):

$$(F.1) \qquad QUF(W) \equiv QUF = a + bW - cW^2 \qquad b, c > 0$$

where W is defined as in (2.1).

If the function (4.1) has positive first derivative and negative second derivative, it represents a risk-averse person with insatiable appetite, that is:

$$QUF' = b - 2cW > 0 \Rightarrow W < \frac{b}{2c} \equiv W_0(1 + \mu_M)$$
$$QUF'' = -2c < 0 \qquad \Rightarrow c > 0$$

$$ARA[QUF] = -\frac{QUF''}{QUF'} = \frac{2c}{b - 2cW} > 0 \;, \qquad RRA[QUF] = \frac{2cW}{b - 2cW}$$

In the Appendices F, G, H we take into consideration $r \backsim N(\mu, \sigma^2)$.

$W_0(1 + \mu_M)$ is the maximum value allowed for W such that (F.1) maintains its characteristic of Risk aversion.

Proposition F.1: *With the definition* $b = 2cW_0(1 + \mu_M)$, *the expected value of QUF in* (4.1), $E[Q(\mu_M)](\sigma, \mu)$, *is a function of Standard Deviation σ and Expected Return μ represented by a paraboloid in the space* $(\sigma, \mu, E[Q(\mu_M)](\sigma, \mu))$ *with downward concavity, whose vertex is given by the point* $\left(0, \mu_M, E[Q(\mu_M)](0, \mu_M)\right)$*. That is:*

$$E[Q(\mu_M)](\sigma, \mu) = QUF(W_0) + cW_0^2\mu_M^2 - cW_0^2[\sigma^2 + (\mu - \mu_M)^2]$$

where $UF(W_0) = a + bW_0 - cW_0^2 = a + 2cW_0(1 + \mu_M)W_0 - cW_0^2$.

Proof: Consider the expected value of the Quadratic Utility Function (F.1):

$$E[Q(\mu_M)] = E[a + bW - cW^2]$$

$$= E[a + bW_0(1 + r) - cW_0^2(1 + r)^2]$$

$$= a + bW_0(1 + E[r]) - cW_0^2(1 + 2E[r] + E[r^2])$$

$$= a + bW_0 + bW_0\mu - cW_0^2 - 2cW_0^2\mu - cW_0^2(\sigma^2 + \mu^2)$$
$$= QUF(W_0) + W_0\mu(b - 2cW_0) - cW_0^2(\sigma^2 + \mu^2)$$

Substituting parameter *b* with its expression, we have:

$$E[Q(\mu_M)] = QUF(W_0) + W_0\mu(2cW_0 + 2c\mu_M W_0 - 2cW_0) - cW_0^2(\sigma^2 + \mu^2)$$

$$= QUF(W_0) + 2cW_0^2\mu\mu_M - cW_0^2(\sigma^2 + \mu^2)$$

Adding and subtracting the same quantity $cW_0^2\mu_M^2$and considering the $E[Q(\mu_M)]$ as a function of σ and μ we obtain:

$$(F.2) \qquad E[Q(\mu_M)](\sigma,\mu) = QUF(W_0) + cW_0^2\mu_M^2 - cW_0^2[\sigma^2 + (\mu - \mu_M)^2]$$

The expression (F.2) represents a paraboloid in the space $(\sigma, \mu, E[Q(\mu_M)](\sigma,\mu))$ with downward concavity, whose vertex is the point $\left(0, \mu_M, E[Q(\mu_M)](0, \mu_M)\right)$.

We assume for simplicity $W_0 = 1$:

$$E[Q(\mu_M)](\sigma,\mu) = \psi(\sigma,\mu) = QUF(W_0) + c\mu_M^2 - c[\sigma^2 + (\mu - \mu_M)^2]$$

And we have

$$\frac{\partial\psi(\sigma,\mu)}{\partial\mu} = -2c(\mu-\mu_M) \quad , \frac{\partial\psi(\sigma,\mu)}{\partial\sigma} = -2c\sigma$$

that will be used for to compute the (3.6) for the Quadratic Utility Function case.

Appendix G

Case QUF 1: $R(\sigma,\mu) = Value\ at\ Risk = \ VaR(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = \mu$

$\psi(\sigma,\mu) = Expected\ QUF\ \ with\ \ \mu_M = 0.3,\ a = 10,\ b = 3,\ \mathrm{c} = 15.$

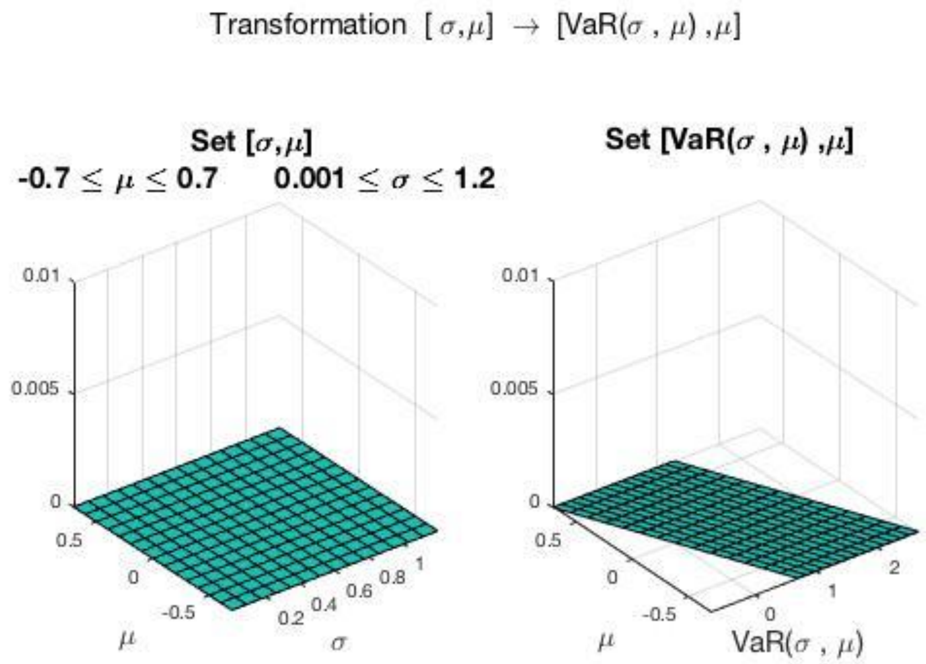
$\alpha \qquad = Confidence\ Level = 0.95$

It is possible to analyze the behavior of $VaR \equiv VaR(\sigma,\mu)$, starting from the transformation:

$$(G.1) \qquad R(\sigma,\mu) = VaR(\sigma,\mu) = -\mu + \sigma\Phi_{-1}(\alpha)\,, \quad T(\sigma,\mu) = \mu$$

The (G.1) transforms the set $[\sigma,\mu]$ in the set $[VaR(\sigma,\mu),\ \mu]$ as is possible to see:

Figure G.1: Transformation $[\boldsymbol{\sigma}, \boldsymbol{\mu}] \rightarrow [\mathbf{VaR}(\boldsymbol{\sigma}, \boldsymbol{\mu}),\ \boldsymbol{\mu}]$



The partial derivatives, using (G.1) are:

$$\frac{\partial \sigma_T}{\partial \sigma} = \Phi_{-1}(\alpha); \;\; \frac{\partial \sigma_T}{\partial \mu} = -1; \;\; \frac{\partial \mu_T}{\partial \mu} = 1; \;\; \frac{\partial \mu_T}{\partial \sigma} = 0;$$

From (3.6), DC1: $2c\sigma > 0$ is true.

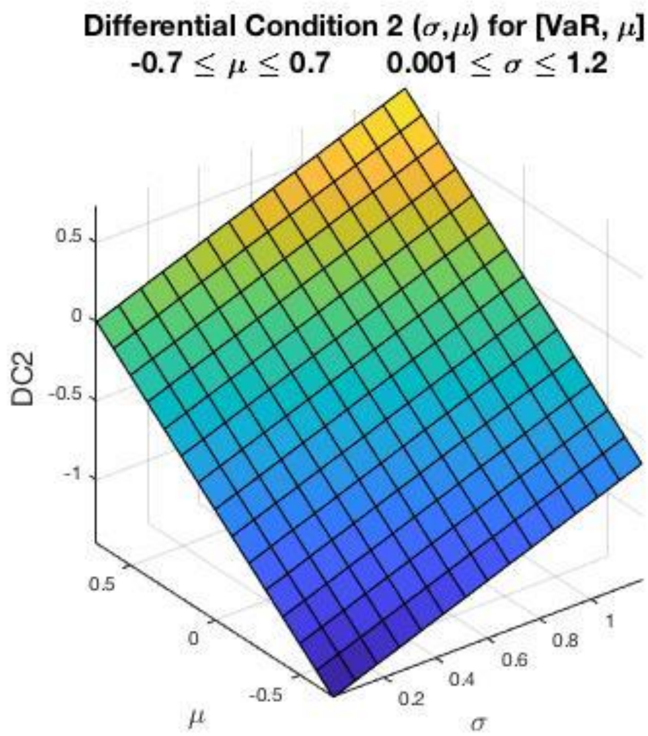
From (3.6), DC2:

$$-2c\Phi_{-1}(\alpha)(\mu - \mu_M) - (-2c\sigma)(-1) > 0$$

$$(G.2) \qquad \frac{\sigma}{\Phi_{-1}(\alpha)} < -(\mu - \mu_M) \rightarrow \frac{\sigma}{\Phi_{-1}(\alpha)} + \mu < \mu_M$$

we can represent the DC2 in closed form:

Figure G.2: Differential Condition2 for $[\mathbf{VaR}(\boldsymbol{\sigma}, \boldsymbol{\mu}),\ \boldsymbol{\mu}]$



DC2 is not satisfied, as is possible to see from the *Figure G.1*. This means that this transformation, even if based on the *Risk Averse Utility Function*, does not preserve the characteristic of the concavity and there are regions in the domain where:

$$\frac{\partial\psi}{\partial T} = \frac{\partial\psi}{\partial\mu} < 0$$

that is not typical of the *Risk Averse Utility Function, Theorem 2.1*.
From (G.2), taking in consideration that:

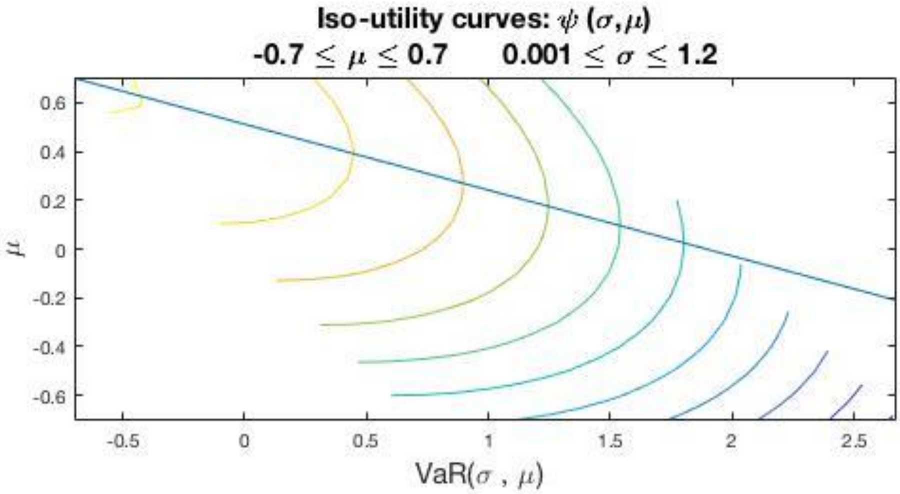
$$\frac{VaR + \mu}{\Phi_{-1}(\alpha)} = \sigma$$

we have:

$$(G.3) \qquad \frac{VaR + \mu}{[\Phi_{-1}(\alpha)]^2} + \mu < \mu_M \;\rightarrow\; \mu(1 + [\Phi_{-1}(\alpha)]^2) < -VaR + \mu_M[\Phi_{-1}(\alpha)]^2$$

The DC2 is respected only below the straight line (G.3), above the straight-line the iso-utility curves have negative slope.
From (3.6), DC3: $\Phi_{-1}(\alpha) > 0$ is true.

Figure G.3: Iso-utility curves of $\psi(\sigma, \mu)$ in 2D $[VaR(\sigma, \mu), \mu]$



Appendix H

Case QUF 2: $R(\sigma,\mu) = Value\ at\ Risk = \ ES(\sigma,\mu)$

$T(\sigma,\mu) = Expected\ Return = \mu$

$\psi(\sigma,\mu) = Expected\ QUF\ \ with\ \ \mu_M = 0.3,\ a = 10,\ b = 3,\ \mathrm{c} = 15.$

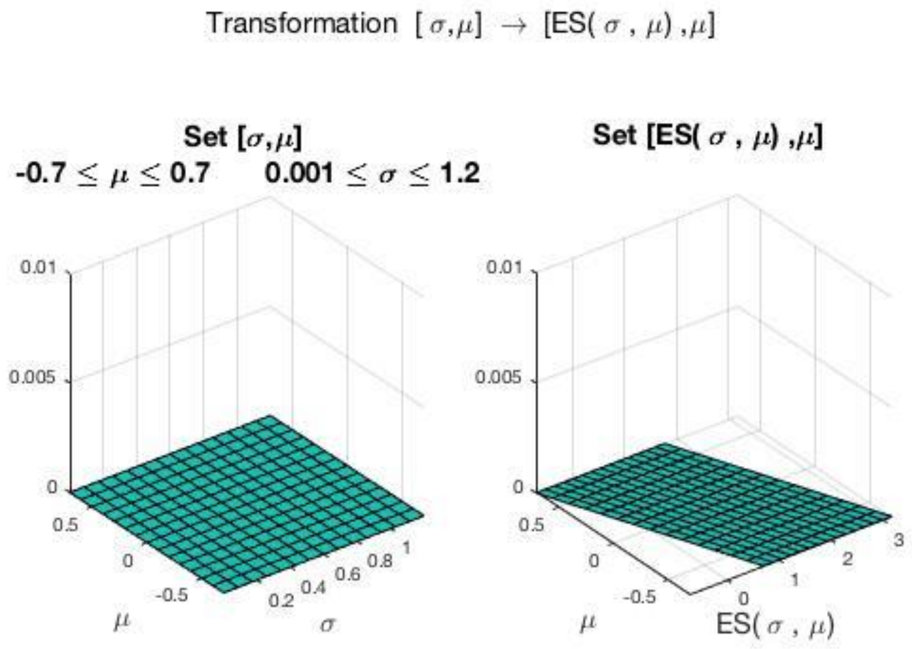
$\alpha \qquad = Confidence\ Level = 0.95$

It is possible to analyze the behavior of $ES \equiv ES(\sigma,\mu)$, starting from the transformation:

$$(H.1) \qquad R(\sigma,\mu) = ES(\sigma,\mu) = -\mu + \frac{\sigma}{1-\alpha}\phi[\Phi_{-1}(\alpha)]\,, \quad T(\sigma,\mu) = \mu$$

The (H.1) transforms the set $[\sigma,\mu]$ in the set $[ES(\sigma,\mu),\ \mu]$ as is possible to see:

Figure H.1: Transformation $[\sigma, \mu] \rightarrow [ES(\sigma, \mu),\ \mu]$



From (3.6), DC1: $2c\sigma > 0$ is true.

From (3.6), DC2:

$$-2c\frac{\phi[\Phi_{-1}(\alpha)]}{1-\alpha}(\mu-\mu_M)-(-2c\sigma)(-1)>0$$

$$(H.2) \qquad \frac{\sigma(1-\alpha)}{\phi[\Phi_{-1}(\alpha)]}<-(\mu-\mu_M)\rightarrow\frac{\sigma(1-\alpha)}{\phi[\Phi_{-1}(\alpha)]}+\mu<\mu_M$$

We can represent the DC2 in closed form:

DC2 is not satisfied, as it is possible to see from the *Figure H.2*. This means that this transformation, even if it is based on the *Risk Averse Utility Function*, does not preserve the characteristic of the concavity and there are regions in the domain where

$$\frac{\partial\psi}{\partial T}=\frac{\partial\psi}{\partial\mu}<0$$

that is not typical of the *Risk Averse Utility Function (Theorem2.1).*

From (H.2), taking into consideration that:

$$\frac{(ES+\mu)(1-\alpha)}{\phi[\Phi_{-1}(\alpha)]}=\sigma$$

we have:

$$(H.3) \qquad \frac{(ES+\mu)(1-\alpha)^2}{\phi[\Phi_{-1}(\alpha)]^2}+\mu<\mu_M\ \rightarrow\ \mu\{(1-\alpha)^2+\phi[\Phi_{-1}(\alpha)]^2\}<-ES+\mu_M\frac{\phi[\Phi_{-1}(\alpha)]^2}{(1-\alpha)^2}$$

The DC2 is respected only below the straight line (H.3), above the straight-line the iso-utility curves have negative slope.

Figure H.2: Differential Condition2 for $[ES(\sigma, \mu),\ \mu]$

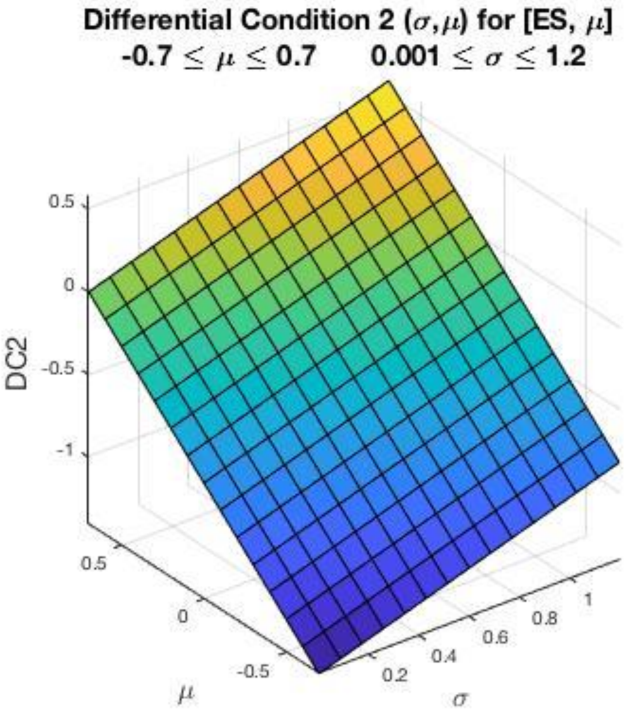


Figure H.3: Iso-utility curves of $\psi(\sigma,\mu)$ in 2D $[ES(\sigma,\mu),\mu]$