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Optimal Stopping Time Strategy for Paying Tax

Soufiene ASSIDI

Department of Accounting and Finance, Higher Institute of Computer Science and Management of Kairouan, Kairouan University, Tunisia

Email: assidisoufiene@yahoo.fr

Noureddine JILANI BEN NAOUARA

Department of Applied Mathematics, Honoris United Universities, ESPRIT

Email: noureddine.jilani@esprit.tn

Wing-Keung Wong

Department of Finance, Quantum AI Research Center, Fintech & Blockchain Research Center, and Big Data Research Center, Asia University, Taiwan;
Department of Medical Research, China Medical University Hospital, Taiwan;
Business, Economic and Public Policy Research Centre, Hong Kong Shue Yan University;
The Economic Growth Centre, Nanyang Technological University, Singapore

Email: wong@asia.edu.tw

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Abstract

Purpose: In this study, we examine the strategic timing for paying corporate taxes under varying financial market conditions.

Design/methodology/approach: We develop a model by using optimal stopping theory and considering three tax regimes: (i) no tax, the firm is never required to pay taxes; (ii) paid tax on a monthly basis, taxes paid based on revenues periodically; and (iii) deferred tax with penalties, delaying taxes but incurring financial penalties.

Findings: Our results indicate that the optimal timing of tax payments is highly sensitive to market parameters such as drift, volatility, and the structure of tax penalties. The evidence further suggests that, under certain risk conditions, tax deferral can serve as a legitimate and value-enhancing strategy, allowing firms to retain more earnings and allocate resources more efficiently.

Originality: Our main contribution is to identify how timing strategies are shaped by drift, volatility, and tax penalties. The model is developed theoretically and supported by a Monte Carlo simulation. We also address the legal and policy implications of tax deferral, namely that under certain risk conditions, tax deferral increases firm value. Overall, our conclusions are based on mathematically rigorous modeling and simulations.

Keywords: Optimal stopping problem, Markov processes, tax payment, reinvestment, penalty.

JEL classification: H26, C61, C63

1. Introduction

In an increasingly volatile global economy, managing tax obligations has become a strategic concern, particularly for small and medium-sized enterprises (SMEs) navigating financial uncertainty. The rigid timing of corporate tax payments,especially in the face of macroeconomic shocks such as inflation, pandemics, or energy crises can strain liquidity, hinder investment, and even threaten firm survival. While tax deferral is a legally permissible strategy that allows companies to delay payments (often subject to penalties), there remains a limited theoretical understanding of when such deferral is optimal under uncertainty. Unlike tax evasion or aggressive avoidance schemes, tax deferral operates within legal frameworks but poses complex decision-making challenges that intersect taxation, accounting, and corporate finance.

Although tax strategy and its implications for firm performance have been widely examined (Hanlon & Heitzman, 2010; Tang & Firth, 2012), most studies rely on empirical or normative perspectives. A growing stream of literature, including work by Marinovic and Varas, (2019) and Badertscher et al., (2023) , has called for more rigorous theoretical modeling of tax-related decisions, particularly under risk. However, the application of optimal stopping theory to tax payment timing remains largely unexplored. This gap is particularly notable given recent interest in using stochastic control models to optimize financial decisions (Bayraktar & Ludkovski, 2011; Gollier, 2013), including taxation. This study addresses this gap by developing a mathematical model based on optimal stopping theory to identify the best timing strategy (τ^*) for deferred tax payment under uncertainty. We model income as a geometric Brownian motion and examine how different tax regimes immediate payment, scheduled tax, and deferral with penalties–impact firm value. In doing so, the study incorporates bounded rationality and decision-making under risk (Simon, 1986; Kahneman et al., 2021) , aligning the model with IFRS (International Financial Reporting Standards), which emphasize fair presentation of tax obligations and deferred tax assets/liabilities (IAS 12). These considerations are crucial for firms reporting under IFRS, as tax timing decisions influence not only cash flow but also financial statement accuracy and investor perceptions (Nobes & Schwencke, 2007).

The paper makes several contributions. First, it provides a quantitative decision-making framework for analyzing optimal tax payment timing under stochastic income paths. Second, it integrates IFRS reporting standards into the theoretical foundation, offering insights on how deferred tax decisions affect both compliance and transparency. Third, it introduces policy implications by demonstrating that moderate deferral, even with penalties, can preserve capital and enhance long-term firm value under certain market conditions. This supports calls for more adaptive tax frameworks, such as time-linked payment smoothing, penalty calibration, and conditional deferral schemes–especially relevant for SMEs and crisis-prone industries.

This research highlights the need for more flexible tax payment regimes from a policy standpoint, especially in nations that use IFRS or are in line with the Base Erosion and Profit Shifting (BEPS) action plans of the OECD. Without jeopardizing tax revenue goals, regulatory tools that enable businesses to proactively control the time of their tax payments while maintaining transparency and compliance can improve financial resilience.

Nevertheless, the study has limitations. The model assumes a simplified tax structure and does not fully account for jurisdiction-specific rules, heterogeneity in firm behavior, or multi-period tax loss carryforward effects (Atwood & Reynolds, 2008; Farhi& Gabaix, 2020). Moreover, it abstracts from dynamic enforcement environments and behavioral anomalies such as managerial risk aversion. Future research could extend this framework to include empirical testing across IFRS-reporting jurisdictions, the role of corporate governance in tax decisions, or interactions between tax timing and financing constraints.

In conclusion, this paper contributes to the intersection of accounting, taxation, decision science, and financial mathematics by modeling the optimal deferral of taxes under uncertainty. It opens new pathways for understanding how timing decisions–rooted in decision theory and aligned with IFRS standards–can support sustainable corporate tax strategies amid economic and regulatory complexity.

The remainder of the paper is organized as follows. Section 2 presents the literature review, Section 3 introduces the mathematical modeling approach, and Section 4 reviews the theory of optimal stopping. Section 5 details the assumptions and main theoretical results. Section 6 presents numerical simulations under different scenarios. Section 7 conclusion and discusses the policy implications.

2. Literature Review and Theoretical Background

This study is based on several root theories in tax and accounting, offering a qualitative approach to studying corporate payment timing of tax as a strategic financial decision guided by market forces and regulatory frameworks.

Pioneering the pack is Optimal Stopping Theory (OST), with a rigorous mathematical framework to characterize decision-making in conditions of uncertainty, enabling firms to come up with the optimal time of payment of taxes in order to optimize firm value and retained earnings (Øksendal & Sulem, 2019; Peskir & Shiryaev, 2006; Assidi et al., 2016) . OST's extensive use in finance and accounting, particularly in investment timing, option valuation, and asset liquidation (Dixit & Pindyck, 1994; De Angelis et al., 2017) , is a testament to its applicability for tax payment decision modeling as issues of timing(Assidi & Omri, 2017).

Building on this, Real Options Theory conceptualizes the deferral of tax payments as a managerial option, where firms exercise flexibility to delay tax obligations until market conditions such as profitability or liquidity improve, thereby optimizing tax liabilities (Copeland & Keenan, 1998; Trigeorgis & Reuer, 2017). This option-like feature gains particular importance in volatile market environments, where timing significantly influences cash flow management and reinvestment opportunities(Assidi & Omri, 2017).

Accounting Conservatism Theory, from a purely accounting perspective, is the theory explaining asymmetric gain and loss recognition, influencing taxable income reporting timing and thus the firm's tax liability over time (Basu, 1997; LaFond & Watts, 2008). This conservatism bias–tendency to recognize losses ahead of time and gains later on–is directly interacting with tax payment timing strategies to affect firmś responses to regulatory and economic conditions (Watts, 2003; Stolowy & Breton, 2004).

Apart from that, the Tax Smoothing Hypothesis further presumes that firms consciously schedule their tax payments to minimize temporal variations so as to stabilize their cash flows and avoid the costs of financing tax payments (Barro, 1979; Auerbach & Hines, 2002; Desai & Dharmapala, 2006). Such behavior rationalizes postponing tax payments even when imposing penalties since firms weigh present costs against future financial stability and investment requirements.

Our study also integrates bounded rationality concepts of decision making and perception of risk (Simon, 1986; Kahneman & Tversky, 1979; Kahneman & Riis, 2005), acknowledging that corporate tax timing decisions are undertaken under informational and cognitive constraints. By including stochastic processes for describing market volatility and drift, our model is able to account for real-world uncertainty with which companies have to contend.

3. Motivation and mathematical modeling

Financial difficulties are faced by many small investors who have difficulty in making both loan repayments and tax obligations, which, in turn, have pushed some small companies to the brink of bankruptcy. To address financial difficulties, this paper explores the potential for financial recovery during such a crisis by emphasizing the strategic advantage of deferring tax payments. To do so, we first discuss the optimal stopping problem in continuous time in the next subsection. In the context of corporate tax planning, the firm's objective is to choose the optimal time τ^* to pay taxes in order to maximize the expected present value of retained income. This problem is modeled as an optimal stopping problem in continuous time, as shown in the following:

$$V(y) = \sup_{\tau \geq 0} \mathbb{E}_y \left[e^{-r\tau} h(Y_\tau) \right], \quad \text{where } h(y) = \max(y, 0) \ . \tag{1}$$

Let's assume that the retained income process Y_t evolves according to the dynamics of the following geometric Brownian motion:

$$dY_t = aY_t \, dt + bY_t \, dB_t \ , \tag{2}$$

where a is the drift, b is the volatility, and B_t is a standard Brownian motion.

Let $\mathcal{L}$ denote the infinitesimal generator associated with Y_t. For a sufficiently smooth function f, the generator acts as shown in the following equation:

$$\mathcal{L}f(y) = ayf'(y) + \frac{1}{2}b^2y^2f''(y) \ , \tag{3}$$

and the value function $V(y)$ satisfies the following variational inequality:

$$\max \{\mathcal{L}V(y) - rV(y), \, h(y) - V(y)\} = 0 \ . \tag{4}$$

This setup defines a free-boundary problem. Let y^* denote the optimal stopping threshold, then we have

- The **continuation region** is $\mathcal{C} = \{y < y^*\}$, where the firm postpones paying taxes, and
- the **stopping region** is $\mathcal{S} = \{y \geq y^*\}$, where the firm chooses to pay taxes immediately.

In the continuation region, we assume a solution of the following form:

$$V(y) = Ay^\lambda \tag{5}$$

where λ is the positive root of the following characteristic equation:

$$\frac{1}{2}b^2\lambda(\lambda - 1) + \mu\lambda - r = 0 \ . \tag{6}$$

To determine constants, we use the following approaches:

- **Value matching:** $A(y^*)^\lambda = y^*$, and
- **Smooth pasting:** $\lambda A(y^*)^{\lambda-1} = 1$.

To solve the above system, we use the following formula:

$$A = \frac{(y^*)^{1-\lambda}}{\lambda}, \quad V(y) = \frac{(y^*)^{1-\lambda}}{\lambda} y^\lambda \quad \text{for } y < y^* \ . \tag{7}$$

In the stopping region $y \geq y^*$, it is optimal to exercise the option and pay the tax by using the following formula:

$$V(y) = cy \, . \tag{8}$$

Equation (8) fully characterizes the value function and optimal policy across both regions. We will discuss how to solve the problem in the next section.

4. Model Formulation and Assumptions

We model the firm's income process X_t by using the following equation:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \ , \tag{9}$$

where μ is the drift function, σ is the volatility function, and B_t is a standard Brownian motion.

4.1 Tax and Penalty Scenarios

Let Y_t denote the retained income of the company. We consider the following three scenarios:

- **Scenario 1: No Tax**
  In this baseline case, no tax or penalty is imposed, so the retained income coincides with the income such that:
  $$Y_t = X_t.$$

- **Scenario 2: Continuous Tax Payment**
  Taxes are paid continuously at a proportional rate $c_2 > 0$. The tax process evolves deterministically as shown in the following:
  $$dT_t = c_2 X_t\,dt,$$
  and the retained income is given by:
  $$Y_t = X_t - T_t.$$

- **Scenario 3: Deferred Tax with Penalty**
  Tax and penalty liabilities are modeled as stochastic processes (T_t) and (P_t) to capture uncertainty and timing effects:
  $$Y_t = X_t - T_t - P_t.$$
  where the penalty process P_t is defined as:
  $$P_t = c_0 t + c_1 T_t,$$
  in which
  - $c_0 > 0$ represents a fixed cost per unit of time delay, and
  - $0 < c_1 < 1$ represents a proportional penalty on the deferred tax amount T_t.

  This setup assumes that taxes are deferred by one period (e.g., one month), and the penalty is applied to the tax due from the prior period.

4.2 Retained Income Process

The taxpayer chooses a stopping time τ to pay tax and penalty liabilities. The retained income process is:

$$Y_t = \begin{cases} X_t, & t < \tau, \\ X_\tau - T_\tau - P_\tau, & t \geq \tau, \end{cases} \tag{10}$$

in which both T_t and P_t depend on the following scenarios:

- Scenario 1: $T_t = 0$ and $P_t = 0$,
- Scenario 2: $T_t = \int_0^t c_2 X_s \, ds$ and $P_t = 0$, and
- Scenario 3: T_t and P_t are defined in the above.

Remark 1 *In the rest of the paper, unless otherwise stated, we will denote Y_t as the retained income under the scenario being analyzed; that is,*

$$Y_t = \begin{cases} X_t & \textit{(Scenario 1: No Tax)}, \\ X_t - T_t & \textit{(Scenario 2: Monthly Paid Tax), and} \\ X_t - T_t - P_t & \textit{(Scenario 3: Deferred Tax with Penalty)}. \end{cases}$$

Each section or simulation will specify which scenario is being considered, and Y_t should be interpreted accordingly.

4.3 Optimal Stopping Problem

The firm aims to choose an optimal stopping time τ^* that maximizes its expected discounted retained income. The problem is formulated as:

$$V(y) = \sup_{\tau \geq 0} \mathbb{E}_y \left[e^{-r\tau} h(Y_\tau) \right], \tag{11}$$

in which

- $r > 0$ is the discount rate,
- $h(y) = \max(y, 0)$ is the reward function, and
- $Y_0 = y$ is the initial retained capital.

We analyze this problem under all three tax scenarios and solve it by using both theoretical and numerical methods.

4.4 Assumptions

We introduce all essential assumptions to underpin our analysis in this section. To do so, we first assume $\lim_{t \to +\infty} e^{-rt} Y_t = 0$ to ensure that $V < +\infty$. We then state the following assumptions:

Assumption A.1 Growth Condition: There exists $K_1 > 0$ such that for all $x \in \mathcal{I}$,

$$\mu^2(x) + \sigma^2(x) < K_1^2(1 + x^2) \ ;$$

Assumption A.2 Global Lipschitz Conditions: There exists a constant $K_2 > 0$ such that, for all $x_1, x_2 \in \mathcal{I}$,

$$|\mu(x_1) - \mu(x_2)| + |\sigma(x_1) - \sigma(x_2)| < K_2 |x_1 - x_2| \ ;$$

Assumption A.3 Volatility Condition: $\sigma^2(x) > 0$ for all $x \geq a$;

Assumption A.4 Drift Condition: $\mu'(x) < r$ for all $x \geq a$ and $\mu(a) \leq 0$; and

Assumption A.5 Drift and Volatility Condition: $\frac{2\mu(y)}{\sigma^2(y)} > 1$ for all $y \geq 0$.

We also assume that the infinitesimal process operator associated with (X_t) is $\mathcal{A}$, defined for any function $h(x)$ with $h \in \mathcal{C}^2(\mathbb{R})$ by the following equation:

$$\mathcal{A}h(x) = \frac{1}{2}\sigma^2(x)h''(.) + \mu(x)h' \ . \tag{12}$$

5. Optimal Stopping in Discrete and Continuous Time

Optimal stopping theory addresses the problem of choosing the optimal time to take a specific action to maximize an expected reward or minimize a cost. In this section, we review the foundational concepts of optimal stopping in both discrete and continuous time, which form the theoretical basis for our subsequent analysis.

5.1 Optimal Stopping in Discrete Time

In a discrete-time framework, let $\{Z_n\}_{n=0}^N$ be an adapted sequence of real-valued random variables defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{0\leq n\leq N}, \mathbb{P})$. The goal is to find a stopping time $\tau^* \in \mathcal{S}_N = \{0, 1, \ldots, N\}$ that maximizes the expected value value as shown in the following equation:

$$V_0 = \sup_{\tau\in\mathcal{S}_N} \mathbb{E}[Z_\tau]. \tag{13}$$

On the other hand, the value function can be computed using the principle of dynamic programming. To do so, we define a backward recursion as shown in the following equation:

$$V_N = Z_N, \tag{14}$$

$$V_n = \max\left(Z_n, \mathbb{E}[V_{n+1} \mid \mathcal{F}_n]\right), \quad \text{for } n = N-1, \ldots, 0. \tag{15}$$

In this model setting, one can easily show that the optimal stopping time τ^* is given by the following equation:

$$\tau^* = \inf\{n \leq N \,:\, V_n = Z_n\}. \tag{16}$$

5.2 Optimal Stopping in Continuous Time

To solve the problem of optimal stopping in continuous time, we let $\{X_t\}_{t\geq 0}$ be a one-dimensional diffusion process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and let $r > 0$ be a discount rate. The problem consists in finding a stopping time $\tau^* \in \mathcal{S}$ such that the value function:

$$V(x) = \sup_{\tau\in\mathcal{S}} \mathbb{E}_x\left[e^{-r\tau}h(X_\tau)\right] \ , \tag{17}$$

is maximized, where $X_0 = x$ and $h(\cdot)$ is a measurable reward function. The stopping region is defined as:

$$\Gamma = \{x \in I \,:\, V(x) = h(x)\}, \tag{18}$$

and the optimal stopping time is characterized by:

$$\tau^* = \inf\{t \geq 0 \,:\, X_t \in \Gamma\}. \tag{19}$$

Under suitable regularity conditions, the value function $V(x)$ in Equation (17) satisfies the following variational inequality:

$$\max\{\mathcal{A}V(x) - rV(x), h(x) - V(x)\} = 0, \tag{20}$$

where $\mathcal{A}$ is the infinitesimal generator of the process X_t, defined as:

$$\mathcal{A}f(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x). \tag{21}$$

Here, $\mu(x)$ and $\sigma(x)$ represent the drift and volatility functions of the diffusion process, respectively.

Under the above model setting, we obtain the following proposition:

Proposition 1 *[Boundary Behavior:] Let $I = (a, b)$ be the state space of X_t. Assuming that both a and b are natural boundaries, and that the reward function $h(x)$ is continuous and satisfies certain integrability conditions, it can be shown (see Dayanik & Karatzas, 2003) that the optimal stopping strategy and value function exist and are unique.*

The continuous framework shown in Proposition 1 forms the theoretical underpinning of our tax timing problem, which will be solved numerically in Section 6 by using Monte Carlo simulations under various taxation scenarios.

6. Results

This section presents the theoretical and simulation-based results derived from our optimal stopping model for tax payment strategies. Our goal is to analyze how different economic and policy parameters such as volatility, drift, tax rates, and penalties affect the firm's decision to defer or pay taxes. The results confirm that under specific conditions, delayed tax payment can increase expected retained income. We first present the theoretical Results obtained in our paper for Geometric Brownian Motion in the next subsection.

6.1 Theoretical Results for Geometric Brownian Motion (GBM)

We first analyze the optimal stopping problem defined in (1) under the assumption that the taxable income process X_t follows the following Geometric Brownian Motion (GBM):

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t, \quad X_0 = x > 0. \tag{22}$$

The solution to the SDE defined in Equation (22) is given by the explicit expression as shown in the following:

$$X_t = X_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}. \tag{23}$$

Under standard assumptions (A.1 to A.5), we show that there exists a solution value function $V(y)$ given by the following proposition:

Proposition 2 (Scenario 1: No Tax) *Under the assumption that no tax is imposed (i.e., $Y_t = X_t$), and if X_t follows a geometric Brownian motion, the value function $V(y)$ associated with the optimal stopping problem satisfies:*

$$V(y) = \begin{cases} d_0^{(1)} y^{\lambda}, & \text{if } y < y_1^*, \\ d_1^{(1)} y, & \text{if } y \geq y_1^*, \end{cases}$$

where $\lambda > 1$ is the positive root of the characteristic equation:

$$\frac{1}{2}\sigma^2\lambda(\lambda-1)+\mu\lambda-r=0,$$

and the constant $d_0^{(1)}$, $d_1^{(1)}$ and threshold y_1^* are determined using smooth-pasting conditions.

Proposition 3 (Scenario 2: Monthly Paid Tax) *When taxes are paid continuously at a proportional rate c_2, i.e., $Y_t = X_t - T_t$ with $dT_t = c_2 X_t dt$, the value function is modified as:*

$$V(y)=\begin{cases} d_0^{(2)} y^{\lambda_2}, & \text{if } y < y_2^*, \\ d_1^{(2)} y, & \text{if } y \geq y_2^*, \end{cases}$$

with an adjusted characteristic equation:

$$\frac{1}{2}\sigma^2\lambda_2(\lambda_2-1)+(\mu-c_2)\lambda_2-r=0,$$

and the constants $d_0^{(2)}$, $d_1^{(2)}$ and y_2^* determined as before.

Proposition 4 (Scenario 3: Deferred Tax with Penalty) *In the case where tax is deferred and a penalty is applied, such that $Y_t = X_t - T_t - P_t$ and $P_t = c_0 t + c_1 T_t$, the value function still satisfies a piecewise structure:*

$$V(y)=\begin{cases} d_0^{(3)} y^{\lambda_3}, & \text{if } y < y_3^*, \\ d_1^{(3)} y + d_2, & \text{if } y \geq y_3^*, \end{cases}$$

where λ_3 is the solution of a characteristic equation incorporating the net return $(\mu - c_2(1+c_1))$:

$$\frac{1}{2}\sigma^2\lambda_3(\lambda_3-1)+(\mu-c_2(1+c_1))\lambda_3-r=0,$$

in which the constants $d_0^{(3)}$, $d_1^{(3)}$, d_2 and y_3^* are determined by value-matching and smooth-pasting.

Proof 1 (Proof of Proposition (Scenario 3: Deferred Tax with Penalty)) *We consider the retained income process:*

$$Y_t = X_t - T_t - P_t,$$

where:

- *$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t$ is a geometric Brownian motion;*
- *$T_t = \int_0^t c_2 X_s\,ds$ is the cumulative tax paid; and*
- *$P_t = c_0 t + c_1 T_t = c_0 t + c_1 \int_0^t c_2 X_s\,ds$ is the penalty process.*

The firmś objective is to choose a stopping time τ that maximizes the expected discounted retained income:

$$V(y)=\sup_{\tau\geq 0}\mathbb{E}_y\left[e^{-r\tau}\max(Y_\tau,0)\right].$$

Step 1: Effective dynamics.
We approximate the retained income process Y_t by aggregating the tax and penalty terms into an effective drift:

$$Y_t \approx X_t - (1 + c_1) \int_0^t c_2 X_s \, ds - c_0 t.$$

To simplify, we define a shifted process $\widetilde{Y}_t = Y_t + c_0 t$, which satisfies approximately:

$$d\widetilde{Y}_t \approx (\mu - c_2(1 + c_1))\widetilde{Y}_t \, dt + \sigma \widetilde{Y}_t \, dB_t.$$

This reduces the problem to a geometric Brownian motion with effective drift $\mu_{\mathit{eff}} = \mu - c_2(1 + c_1)$.

Step 2: HJB equation.
Let $\mathcal{L}$ denote the infinitesimal generator associated with Y_t. Then, in the continuation region $\{y < y_3^*\}$, the value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\max \{\mathcal{L}V(y) - rV(y),\ h(y) - V(y)\} = 0,$$

with reward function $h(y) = \max(y, 0)$ and

$$\mathcal{L}V(y) = (\mu - c_2(1 + c_1))yV'(y) + \frac{1}{2}\sigma^2 y^2 V''(y).$$

In the continuation region, the HJB equation becomes:

$$\frac{1}{2}\sigma^2 y^2 V''(y) + (\mu - c_2(1 + c_1))yV'(y) - rV(y) = 0.$$

Step 3: Solution in the continuation region.
We seek a solution of the form $V(y) = d_0 y^\lambda$. Substituting into the HJB equation yields:

$$\left[\frac{1}{2}\sigma^2 \lambda(\lambda - 1) + (\mu - c_2(1 + c_1))\lambda - r\right] d_0 y^\lambda = 0.$$

Hence, the characteristic equation is:

$$\frac{1}{2}\sigma^2 \lambda(\lambda - 1) + (\mu - c_2(1 + c_1))\lambda - r = 0.$$

Under Assumption A.5, this equation admits a unique positive root $\lambda > 1$, ensuring concavity of $V(y)$ in this region.

Step 4: Solution in the stopping region.
When $y \geq y_3^*$, the firm pays tax and penalty, and thus receives a linear payoff:

$$V(y) = d_1^3 y + d_2.$$

Step 5: Smooth-pasting conditions.
At the free boundary $y = y_3^*$, we impose continuity and differentiability of $V(y)$:

$$\begin{cases} d_0^{(3)}(y^*)^\lambda = d_1^{(3)}y^* + d_2, & \textit{(value matching)} \\ \lambda d_0^{(3)}(y^*)^{\lambda-1} = d_1^{(3)}, & \textit{(smooth pasting)} \end{cases}$$

These two equations allow for determination of $d_0^{(3)}$, $d_1^{(3)}$, d_2, and y_3^* (numerically, if necessary).

From the propositions we have developed in this paper, we get the following remark:

Remark 2 *Scenarios 1 and 2 are special cases of the above proofs because:*

- *In Scenario 1 (No Tax), set $c_2 = c_1 = c_0 = 0$, so, we obtain $\mu_{eff} = \mu$; and*
- *in Scenario 2 (Monthly Tax), set $c_1 = c_0 = 0$, so, we have $\mu_{eff} = \mu - c_2$.*

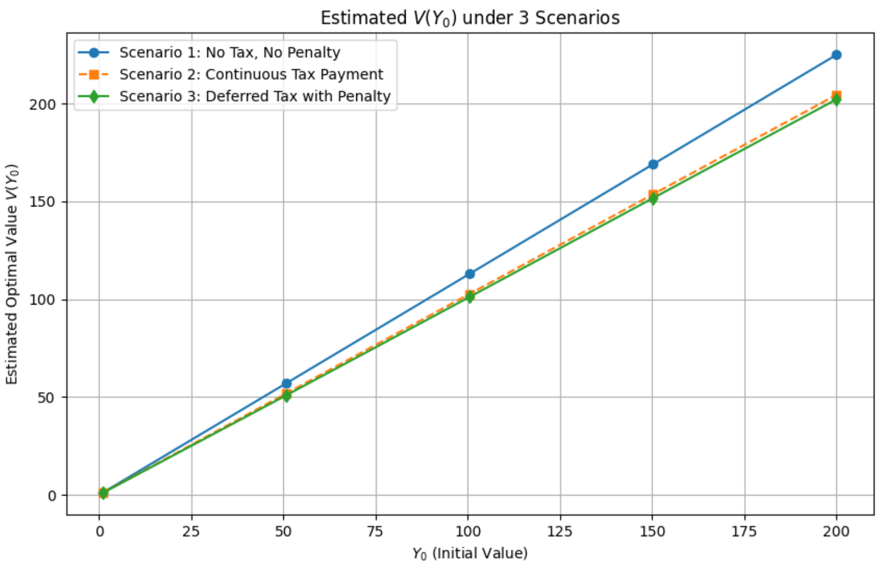
6.2 Simulation-Based Results

We turn to discuss the simulation-based results by using the propositions we have developed in this paper.

First, by using Monte Carlo simulations, we quantify the impact of the model parameters on $V(y)$ and the optimal stopping time τ^*. In our simulation, each experiment involves simulating $N = 1500$ paths with varying inputs. The most significant findings are presented in the following subsection.

6.2.1 Initial Income Y_0

Figure 1: Optimal retained value $V(Y_0)$ as a function of initial capital Y_0.



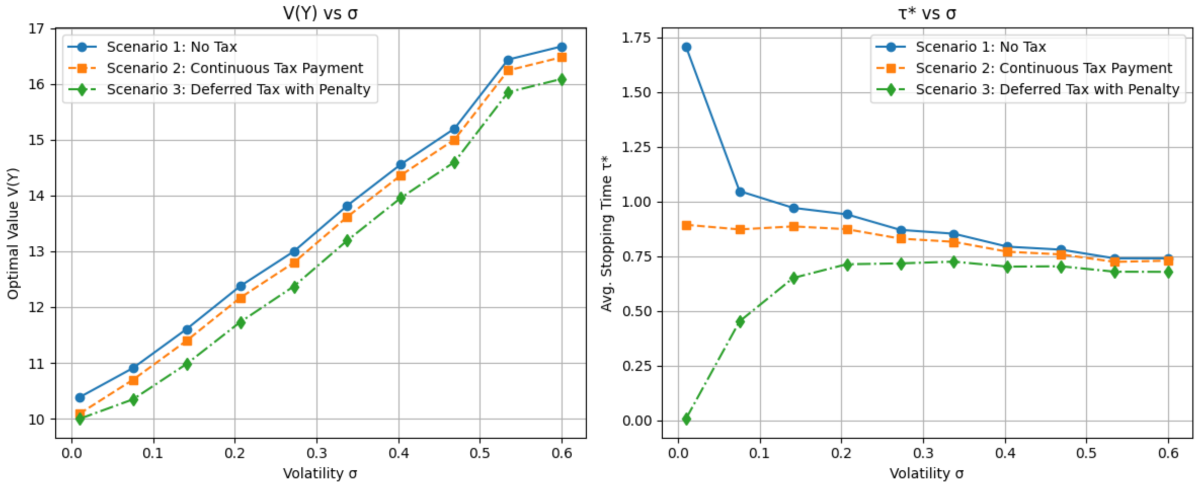
Note: Firms with higher initial capital enjoy greater retained value and are more resilient to penalties. Lower values of Y_0 increase the likelihood of early stopping due to bankruptcy or tax trigger.

From the discussion above, we obtain the following remark:

Remark 3 *It is now evident that the optimal values with and without tax exhibit linearity within certain state intervals. This observation aligns with our theoretical expectations given by Propositions 2, 3, and 4.*

6.2.2 Volatility σ

Figure 2: Effect of volatility σ on optimal value $V(Y_0 = 10)$ (left) and average stopping time τ^* (right) across three tax scenarios.



Note : we observe that increasing σ reduces $V(y)$ and shortens τ^*, especially in deferred tax scenarios. This aligns with risk-averse behavior and validates the concave structure of $V(y)$.

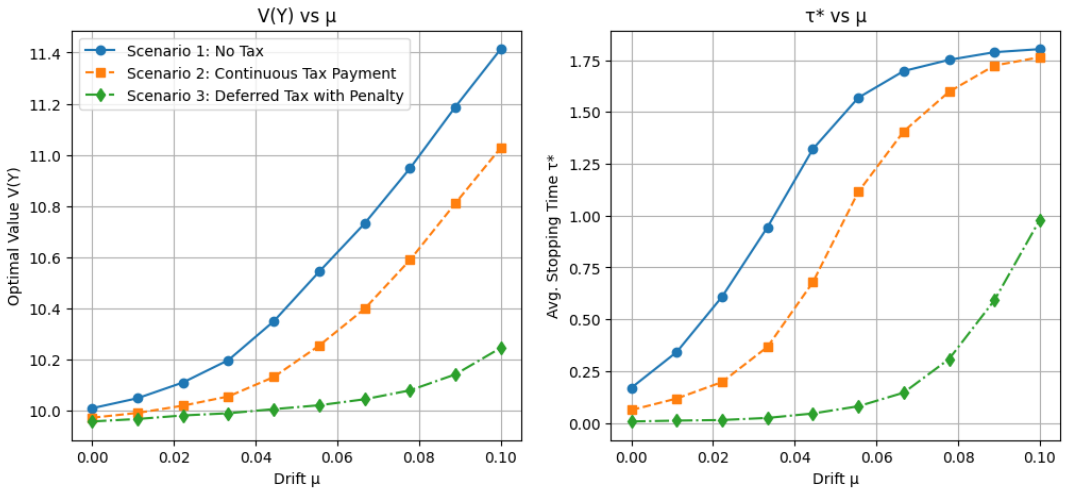
From the theory we developed in this paper, we observe the following remark:

Remark 4 *Under high market volatility and risk increases, firms prefer to pay taxes earlier to avoid losses. This is reflected in a sharp decrease in* τ^* *for the deferred tax case. In addition, our simulations show after some volatility that:*

$$\sigma \uparrow \quad \Rightarrow \quad \tau^* \downarrow, \quad V(y) \uparrow \ .$$

6.2.3 Drift Coefficient μ

Figure 3: Effect of drift μ on optimal value $V(Y_0 = 10)$ (left) and average stopping time τ^* (right) across three tax scenarios.



Note : Higher values of μ increase the expected future income, thereby extending the optimal stopping time τ^*. However, under the taxation with penalty regime, this effect is dampened due to the accumulating penalty. This observation is consistent with the theoretical threshold $y_i^*,\ i = 1, 2, 3$ derived earlier.

Cross Analysis of Figures 3 and 2

The ratio $\frac{2\mu}{\sigma^2}$ plays a central role in evaluating the financial risk level of small and medium-sized enterprises (SMEs). It serves as a useful policy indicator to guide fiscal and financial decisions based on market conditions.

Figure 2 (Impact of volatility σ) : Conversely, as the volatility σ increases (with constant μ), the ratio $\frac{2\mu}{\sigma^2}$ decreases, indicating a higher level of risk for SMEs. In Scenario 3, we observe an initial increase in the optimal stopping time τ^*, followed by a decline as σ becomes too large. This reflects that excessive uncertainty forces firms to act more quickly, even when tax deferral is available.

Figure 3 (Impact of drift μ) : As the drift μ increases (with σ held constant), the ratio $\frac{2\mu}{\sigma^2}$ increases as well. This implies that SMEs become gradually less risky. In this case, Scenario 3 (Deferred Tax with Penalty) becomes increasingly advantageous, especially when $\frac{2\mu}{\sigma^2} > 1$. This highlights the benefit of allowing tax deferral in high-growth contexts to give SMEs the opportunity to stabilize.

Policy Implications:

- **If $\frac{2\mu}{\sigma^2} > 1$:** Firms have a real opportunity to recover. It is in the government's interest to allow SMEs to defer tax payments by reducing the penalty parameter c_0. This grants them additional time to rebuild market position.
- **If $\frac{2\mu}{\sigma^2} \leq 1$:** Firms are highly vulnerable, and deferring taxes alone is not sufficient. In this case, direct government intervention is needed, such as offering interest-free loans or subsidies. These measures effectively increase the drift μ, improving long-term survival chances.

Summary of Figures 2 and 3 The two figures demonstrate that the ratio $\frac{2\mu}{\sigma^2}$ is a reliable decision-making tool to adapt fiscal and financial support policies for SMEs. By adjusting either the tax penalty parameter c_0 or enhancing the growth dynamics through μ, governments can reduce systemic risk and foster business resilience under varying market conditions.

6.2.4 Fixed Penalty Rate c_0

Table 1: Effect of fixed penalty c_0 on $V(Y_0 = 10)$ and average stopping time τ^* (Scenario 3 only)

c_0	V (Scenario 3)	τ^* (Scenario 3)
0.100	11.7309	0.7906
2.575	10.5197	0.2456
5.050	10.1766	0.0833
7.525	10.0780	0.0269
10.000	9.9850	0.00959

Note: In Scenario 3, higher values of c_0 significantly reduce the benefit of delaying taxes. The value function $V(y)$ under deferred tax decreases as c_0 increases, and the optimal stopping time τ^* converges to zero.

6.2.5 Proportional Penalty Rate c_1

Table 2: Effect of Proportional Penalty Rate c_1 on $V(Y_0 = 10)$ and average stopping time τ^* in Scenario 3

c_1	V_3 (Scenario 3)	τ_3^*
0.100	10.0096	0.0568
0.325	10.0063	0.0499
0.550	10.0018	0.0375
0.775	9.9965	0.0350
1.000	9.9887	0.0284

Note: In Scenario 3, where penalties are both fixed and proportional, increasing c_1 reduces the value function $V(y)$ and accelerates the optimal stopping time τ_3^* toward zero, making immediate payment optimal.

Remark 5 *The deferred tax strategy is only advantageous if penalty rates are sufficiently low. When c_0 or c_1 increase, the cost of deferral quickly outweighs the reinvestment benefit:*

$$c_0, c_1 \uparrow \quad \Rightarrow \quad V_{deferred}(y) \downarrow, \quad \tau^* \downarrow$$

6.2.6 Monthly Tax Rate c_2

Table 3: Effect of monthly tax rate c_2 on $V(Y_0 = 10)$ and average stopping time τ^* across three tax scenarios

c_2	V_1 (Scenario 1)	V_2 (Scenario 2)	V_3 (Scenario 3)	τ_1^*	τ_2^*	τ_3^*
0.000	10.4848	10.4848	10.0556	1.5753	1.5753	0.2184
0.025	10.4797	10.1549	10.0067	1.5669	0.8190	0.0453
0.050	10.4736	10.0147	9.9731	1.5715	0.2060	0.0162
0.075	10.4761	9.9662	9.9482	1.5526	0.0550	0.0053
0.100	10.4822	9.9392	9.9264	1.5730	0.0177	0.0025

An increase in the tax rate c_2 lowers the retained income under both the continuous monthly tax (Scenario 2) and the deferred tax with penalty (Scenario 3) regimes, reducing the optimal expected value $V(y)$

Remark 6 *Higher tax rates c_2 reduce the retained value and lead to earlier stopping times in Scenarios 2 and 3, while Scenario 1 remains unchanged.*

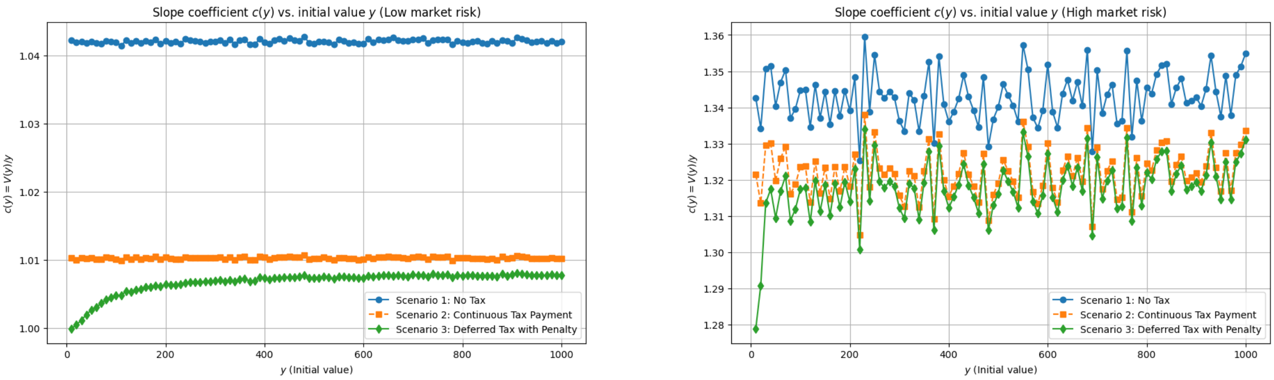
$$c_2 \uparrow \quad \Rightarrow \quad V_{tax}(y) \downarrow, \quad V_{deferred}(y) \downarrow, \quad \tau^* \downarrow$$

6.3 Slope Coefficient and Linearity Verification

To numerically validate the linearity of $V(y)$ in the interval $(0, b^*)$, we compute the slope ratio:

$$c(y) = \frac{V(y)}{y}, \quad y \in (0, y_i^*). \quad i = 1, 2, 3 \tag{24}$$

Figure 4: Slope coefficient $c(y)$ under low and high market risk conditions



Note: The left figure shows the slope coefficient $c(y)$ in a low-risk scenario where $c(y)$ is nearly constant. The right figure shows high-risk conditions, where $c(y)$ becomes volatile, indicating distortion in expected gains.

Now, let's provide the mathematical description of c for each case:

- For reinvestment without tax, where the capital retained by the company follows $dY_t = \mu * Y_t dt + \sigma * Y_t dB_t$, we have:

$$\mathbb{E}_y[e^{-rt}(Y_t)^+] = ye^{(\mu-r)t}$$

  Additionally, $c = e^{(\mu-r)T}$.

- For reinvestment with monthly paid tax, where the income process follows $dY_t = \mu * X_t dt + \sigma * X_t dB_t - dT_t$ with $dT_t = c_2 * X_t dt$, we have:

$$\mathbb{E}_y[e^{-rt}(Y_t)^+] = y(1-c_2)e^{(\mu-r)t}$$

  Here, $c = (1-c_2)e^{(\mu-r)T}$.

- For reinvestment with late tax payment, where the income process follows $dY_t = \mu * X_t dt + \sigma * X_t dB_t - dT_t - dP_t$, we have:

$$\mathbb{E}_y[e^{-rt}(Y_t)^+] = y(1-c_2(1+c_1))e^{(\mu-r)t} - c_0 \sum_{t<T} \mathbb{P}_y\left(dB_t > \frac{(0.5\sigma^2-\mu)}{\sigma}dt\right)$$

  Furthermore, there exists b^* such that $V''(b^*) < 0$, leading to:

$$V(y) = \begin{cases} c * \varphi(y) & \text{for } 0 < y < b^* \\ d_0 * y + d_1 & \text{for } y > b^* \end{cases}$$

  Where:

$$
\begin{aligned}
d_0 &= (1 - c_2(1 + c_1))e^{(\mu-r)T} \\
d_1 &= -\frac{c_0}{2}e^{(\mu-r)T}\left(1 + erf\left(\frac{(\mu - 0.5\sigma^2)dt}{\sigma\sqrt{2dt}}\right)\right) \\
c &= d_0 - \frac{d_1}{y^*}
\end{aligned}
$$

6.4 Optimal stopping times

The optimal stopping time τ^* which represents the decision point at which the firm chooses to pay the tax and associated penalty in order to maximize the expected discounted retained income given in (11)

Under the structure of our model, the optimal stopping strategy is of threshold type. That is, there exists a critical value b^* such that it is optimal to stop (i.e., pay the tax and penalty) when $Y_t \geq b^*$. The stopping time is thus given by:

$$\tau^* = \inf\{t \geq 0 : Y_t \geq b^*\}.$$

Here, b^* represents the optimal stopping boundary that characterizes the firm's behavior under each scenario. Specifically,

$$b^* = \begin{cases} y_1^*, & \text{in Scenario 1 (No Tax)}, \\ y_2^*, & \text{in Scenario 2 (Monthly Paid Tax)}, \\ y_3^*, & \text{in Scenario 3 (Deferred Tax with Penalty)}. \end{cases}$$

Numerical behavior.
Our Monte Carlo simulations confirm the structure above. The average optimal stopping times τ_i^* are sensitive to parameters such as:

- **Volatility σ:** Higher σ increases risk and reduces τ_3^*, making early stopping optimal in the presence of penalties.
- **Penalty parameters (c_0, c_1):** Larger values penalize delay more harshly, reducing τ_3^*.
- **Drift μ:** Higher μ encourages waiting (larger τ^*), unless penalties accumulate too fast.

Summary of stopping time choice
The optimal stopping time τ^* can be explicitly computed or numerically approximated depending on the scenario. This structure is consistent across all three tax regimes considered in this study.

7. Conclusion and policy implications

This paper presents a unified framework of optimal timing of business taxation in optimal stopping theory under various market environments. Through the development of three alternative tax regimes no-tax, taxation with monthly payment, and delayed taxation with penalty.We demonstrate that optimal timing of tax payment significantly relies on market parameters, including drift, volatility, and penalty structure of taxation. Our findings suggest that deferral of tax can be a legitimate and value-enhancing action under some risk environments, enabling companies to maximize earnings retained with better resource allocation.

Theoretical analysis supported by Monte Carlo simulations highlights that those firms having good market conditions can benefit from tax deferral, while firms facing higher volatility may find it optimal to pay taxes earlier to contain growing penalty costs. Such findings necessitate responsive and adaptive tax policies merging firm-level incentives with public finance considerations.

Policy-wise, the study highlights the critical role penalty regimes play in shaping corporate tax behavior. Designing tax regimes with stepped-up fixed and proportional penalties can prevent undue deferral without starving firms of strategic room for maneuvering in order to meet liquidity and investment issues. Particularly during periods of economic crises or recession, temporary deferment of taxes or differential rates of penalties can alleviate financial pressures on firms, reduce insolvency risk, and enhance macroeconomic stability. Policymakers can employ these tools to realize more flexible, stable fiscal conditions that are better aligned with firm incentives and macroeconomic goals.

However, this research has some limitations. The modeling framework simplifies firm-specific heterogeneity, sectoral heterogeneity, and institutional factors that in reality might affect tax payment behavior. The use of stochastic processes and simulation, although strong, might not include all aspects of reality, like behavioral bias or regulatory developments. It also assumes rational decision-making under risk, which might not reflect the bounded rationality that is evident in reality.

Future research should be able to capture these shortcomings by incorporating sectoral characteristics, regulatory divergence, and tax compliance behavioral components. Empirical evidence based on company-level observations from a variety of jurisdictions would further enhance the generalization and usefulness of the findings. These innovations could contribute towards improving our understanding of optimal tax timing techniques and help tax policymakers formulate more advanced and effective tax policy.

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A. Python Code for Numerical Simulations

This appendix provides the Python code used to produce the simulation-based results and figures in Section 6 of the paper.

A.1 Effect of Initial Income Y_0 on $V(Y)$

```
import numpy as np
import matplotlib.pyplot as plt

# --- Parameters ---
mu = 0.05
sigma = 0.25
r = 0.03
c2 = 0.02
c0 = 0.5
c1 = 0.2
Y0 = 10

# Simulation settings
T_max = 3
dt = 1 / 12
num_steps = int(T_max / dt)
num_simulations = 2000
time_grid = np.linspace(0, T_max, num_steps)

def simulate_gbm(Y0, mu, sigma, dt, num_simulations, num_steps):
    X = np.zeros((num_simulations, num_steps))
    X[:, 0] = Y0
    for t in range(1, num_steps):
        Z = np.random.randn(num_simulations)
        X[:, t] = X[:, t-1] * np.exp((mu - 0.5 * sigma**2) * dt + sigma * np.sqrt(dt) * Z)
    return X

def simulate_tax(X, c2, dt):
    T = np.zeros_like(X)
    for t in range(1, X.shape[1]):
        T[:, t] = T[:, t-1] + c2 * X[:, t-1] * dt
    return T

def simulate_penalty(T, c0, c1, dt):
    P = np.zeros_like(T)
    for t in range(1, T.shape[1]):
        current_time = t * dt
        P[:, t] = c0 * current_time + c1 * T[:, t]
    return P

def estimate_optimal_value(Y0):
    X = simulate_gbm(Y0, mu, sigma, dt, num_simulations, num_steps)
    T = simulate_tax(X, c2, dt)
    P = simulate_penalty(T, c0, c1, dt)
    discount_factors = np.exp(-r * time_grid)

    V1 = np.mean(np.max(discount_factors * X, axis=1))
    V2 = np.mean(np.max(discount_factors * np.maximum(X - T, 0), axis=1))
    V3 = np.mean(np.max(discount_factors * np.maximum(X - T - P, 0), axis=1))
    return V1, V2, V3
```

```python
Y0_values = np.linspace(10, 1000, 100)
V1_list, V2_list, V3_list = [], [], []

for y0 in Y0_values:
    v1, v2, v3 = estimate_optimal_value(y0)
    V1_list.append(v1)
    V2_list.append(v2)
    V3_list.append(v3)

plt.figure(figsize=(10, 6))
plt.plot(Y0_values, V1_list, 'o-', label='No Tax')
plt.plot(Y0_values, V2_list, 's--', label='Continuous Tax')
plt.plot(Y0_values, V3_list, 'd-', label='Deferred Tax + Penalty')
plt.xlabel('Initial Income $Y_0$')
plt.ylabel('Optimal Value $V(Y_0)$')
plt.legend()
plt.grid(True)
plt.title('Effect of Initial Income on Optimal Value')
plt.show()
```

A.2 Effect of Volatility σ on Optimal Value and Stopping Time

```python
import numpy as np
import matplotlib.pyplot as plt

mu = 0.05
r = 0.03
c2 = 0.02
c0 = 0.5
c1 = 0.2
Y0 = 10
T = 2.0
dt = 1 / 12
num_simulations = 2000
num_steps = int(T / dt)
time_grid = np.linspace(0, T, num_steps)

def simulate_gbm(Y0, mu, sigma, dt, N, steps):
    X = np.zeros((N, steps))
    X[:, 0] = Y0
    for t in range(1, steps):
        Z = np.random.randn(N)
        X[:, t] = X[:, t-1] * np.exp((mu - 0.5 * sigma**2)*dt + sigma * np.sqrt(dt) * Z)
    return X

def simulate_tax(X, c2, dt):
    return c2 * np.cumsum(X[:, :-1] * dt, axis=1)

def simulate_penalty(T, c0, c1, dt):
    steps = T.shape[1]
    t_grid = np.linspace(0, steps * dt, steps)
    return c0 * t_grid[np.newaxis, :] + c1 * T

def estimate_optimal_value_and_stopping(Y0, sigma):
    X = simulate_gbm(Y0, mu, sigma, dt, num_simulations, num_steps)
    T_tax = simulate_tax(X, c2, dt)
```

```
    P = simulate_penalty(T_tax, c0, c1, dt)
    df = np.exp(-r * time_grid[1:])

    Y1 = X[:, 1:]
    Y2 = np.maximum(X[:, 1:] - T_tax, 0)
    Y3 = np.maximum(X[:, 1:] - T_tax - P, 0)

    V1 = np.mean(np.max(Y1 * df, axis=1))
    V2 = np.mean(np.max(Y2 * df, axis=1))
    V3 = np.mean(np.max(Y3 * df, axis=1))

    tau1 = np.mean(np.argmax(Y1 * df, axis=1)) * dt
    tau2 = np.mean(np.argmax(Y2 * df, axis=1)) * dt
    tau3 = np.mean(np.argmax(Y3 * df, axis=1)) * dt

    return V1, V2, V3, tau1, tau2, tau3

sigma_values = np.linspace(0.01, 0.6, 10)
V1_list, V2_list, V3_list = [], [], []
tau1_list, tau2_list, tau3_list = [], [], []

for sigma in sigma_values:
    V1, V2, V3, t1, t2, t3 = estimate_optimal_value_and_stopping(Y0, sigma)
    V1_list.append(V1)
    V2_list.append(V2)
    V3_list.append(V3)
    tau1_list.append(t1)
    tau2_list.append(t2)
    tau3_list.append(t3)

plt.figure(figsize=(12, 5))
plt.subplot(1, 2, 1)
plt.plot(sigma_values, V1_list, 'o-', label='Scenario 1: No Tax')
plt.plot(sigma_values, V2_list, 's--', label='Scenario 2: Continuous Tax Payment')
plt.plot(sigma_values, V3_list, 'd-.', label='Scenario 3: Deferred Tax with Penalty')
plt.xlabel('Volatility ÏČ')
plt.ylabel('Optimal Value V(Y)')
plt.title('V(Y) vs ÏČ')
plt.grid(True)
plt.legend()

plt.subplot(1, 2, 2)
plt.plot(sigma_values, tau1_list, 'o-', label='Scenario 1: No Tax')
plt.plot(sigma_values, tau2_list, 's--', label='Scenario 2: Continuous Tax Payment')
plt.plot(sigma_values, ta3_list, 'd-.', label='Scenario 3: Deferred Tax with Penalty')
plt.xlabel('Volatility ÏČ')
plt.ylabel('Avg. Stopping Time ÏĎ*')
plt.title('ÏĎ* vs ÏČ')
plt.grid(True)
plt.legend()
plt.tight_layout()
plt.show()
```

A.3 Effect of Drift μ on Optimal Value and Stopping Time

```
import numpy as np
import matplotlib.pyplot as plt
```

```
sigma = 0.02
r = 0.03
c2 = 0.02
c0 = 0.5
c1 = 0.1
Y0 = 10
T = 2.0
dt = 1 / 12
num_simulations = 2000
num_steps = int(T / dt)
time_grid = np.linspace(0, T, num_steps)

def simulate_gbm(Y0, mu, sigma, dt, N, steps):
    X = np.zeros((N, steps))
    X[:, 0] = Y0
    for t in range(1, steps):
        Z = np.random.randn(N)
        X[:, t] = X[:, t-1] * np.exp((mu - 0.5 * sigma**2)*dt + sigma * np.sqrt(dt) * Z)
    return X

def simulate_tax(X, c2, dt):
    return c2 * np.cumsum(X[:, :-1] * dt, axis=1)

def simulate_penalty(T, c0, c1, dt):
    steps = T.shape[1]
    t_grid = np.linspace(0, steps * dt, steps)
    return c0 * t_grid[np.newaxis, :] + c1 * T

def estimate_optimal_value_and_stopping(Y0, mu):
    X = simulate_gbm(Y0, mu, sigma, dt, num_simulations, num_steps)
    T_tax = simulate_tax(X, c2, dt)
    P = simulate_penalty(T_tax, c0, c1, dt)
    df = np.exp(-r * time_grid[1:])

    Y1 = X[:, 1:]
    Y2 = np.maximum(X[:, 1:] - T_tax, 0)
    Y3 = np.maximum(X[:, 1:] - T_tax - P, 0)

    V1 = np.mean(np.max(Y1 * df, axis=1))
    V2 = np.mean(np.max(Y2 * df, axis=1))
    V3 = np.mean(np.max(Y3 * df, axis=1))

    tau1 = np.mean(np.argmax(Y1 * df, axis=1)) * dt
    tau2 = np.mean(np.argmax(Y2 * df, axis=1)) * dt
    tau3 = np.mean(np.argmax(Y3 * df, axis=1)) * dt

    return V1, V2, V3, tau1, tau2, tau3

mu_values = np.linspace(0.0, 0.1, 10)
V1_list, V2_list, V3_list = [], [], []
tau1_list, tau2_list, tau3_list = [], [], []

for mu in mu_values:
    V1, V2, V3, t1, t2, t3 = estimate_optimal_value_and_stopping(Y0, mu)
    V1_list.append(V1)
    V2_list.append(V2)
```

```
    V3_list.append(V3)
    tau1_list.append(t1)
    tau2_list.append(t2)
    tau3_list.append(t3)

plt.figure(figsize=(12, 5))
plt.subplot(1, 2, 1)
plt.plot(mu_values, V1_list, 'o-', label='Scenario 1: No Tax')
plt.plot(mu_values, V2_list, 's--', label='Scenario 2: Continuous Tax Payment')
plt.plot(mu_values, V3_list, 'd-.', label='Scenario 3: Deferred Tax with Penalty')
plt.xlabel('Drift Î ')
plt.ylabel('Optimal Value V(Y)')
plt.title('V(Y) vs Î ')
plt.grid(True)
plt.legend()

plt.subplot(1, 2, 2)
plt.plot(mu_values, tau1_list, 'o-', label='Scenario 1: No Tax')
plt.plot(mu_values, tau2_list, 's--', label='Scenario 2: Continuous Tax Payment')
plt.plot(mu_values, tau3_list, 'd-.', label='Scenario 3: Deferred Tax with Penalty')
plt.xlabel('Drift Î ')
plt.ylabel('Avg. Stopping Time ÏĎ*')
plt.title('ÏĎ* vs Î ')
plt.grid(True)
plt.show()
```

A.4 Effect of Penalty Rates c_0 and c_1

```
    # c0 variation
c0_values = np.linspace(0.1, 10, 5)
results_c0 = []

for c0_val in c0_values:
    V1, V2, V3, t1, t2, t3 = estimate_optimal_value_and_stopping(Y0, sigma)
    results_c0.append([c0_val, V1, V2, V3, t1, t2, t3])

# c1 variation
c1_values = np.linspace(0.1, 1, 5)
results_c1 = []

for c1_val in c1_values:
    V1, V2, V3, t1, t2, t3 = estimate_optimal_value_and_stopping(Y0, sigma)
    results_c1.append([c1_val, V1, V2, V3, t1, t2, t3])
```

A.5 Effect of Monthly Tax Rate c_2

```
    c2_values = np.linspace(0.0, 0.1, 5)
results_c2 = []

for c2_val in c2_values:
    V1, V2, V3, t1, t2, t3 = estimate_optimal_value_and_stopping(Y0, mu)
    results_c2.append([c2_val, V1, V2, V3, t1, t2, t3])
```

A.6 Slope Coefficient and Linearity Verification

```
Y0_values = np.linspace(100, 400, 10)
V1_list, V2_list, V3_list = [], [], []
```

```python
for y0 in Y0_values:
    v1, v2, v3 = estimate_optimal_value(y0)
    V1_list.append(v1)
    V2_list.append(v2)
    V3_list.append(v3)

plt.figure(figsize=(10, 6))
plt.plot(Y0_values, np.array(V1_list)/Y0_values, 'o-', label='No Tax')
plt.plot(Y0_values, np.array(V2_list)/Y0_values, 's--', label='Tax Only')
plt.plot(Y0_values, np.array(V3_list)/Y0_values, 'd-', label='Tax + Penalty')
plt.xlabel('$Y_0$')
plt.ylabel('$V(Y_0)/Y_0$')
plt.title('Slope coefficient $c(y)$ vs. $y$')
plt.grid(True)
plt.legend()
plt.show()
```