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Admission controls for Erlang's loss system with service times distributed as a finite sum of exponential random variables

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Abstract. It is known that a threshold policy (or trunk reservation policy) is optimal for Erlang's loss system under certain assumptions. In this paper we examine the robustness of this policy under departures from the standard assumption of exponential service times (call holding times) and give examples where the optimal policy has a generalized trunk reservation form.

Keywords: MISSING KEYWORDS

1. Introduction

We consider a single link loss network consisting of C circuits or servers, each able to carry a single call or customer. Calls of type $r \in \mathcal{R} = \{1, 2, \ldots, R\}$ arrive as independent Poisson processes at rate λ_r. For ease of exposition, we shall often consider $\mathcal{R} = \{1, 2\}$, but the results we obtain are easily extended to larger numbers of distinct call types. Calls that arrive when all C circuits are in use are lost or blocked. We shall assume that call holding times are distributed as the sum of I independent exponential random variables where the i^{th} variable has mean $1/\mu_i$. Thus we can think of the service time for a customer as being the time it takes for a Markov process to move from states 1 through to I and then to leave state I, assuming that when the process leaves state i it must move to $i + 1$, and that the time it spends in state i is exponentially distributed with mean $1/\mu_i$. When this process is in state i we say the customer is in the i^{th} service phase. The holding time distribution we describe here is a special case of more general phase-type distributions. Finally, we assume that all holding times and interarrival times are independent of one another.

We are interested in the situation where a reward w_r is received whenever a call of type r is accepted. We assume without loss of generality that $w_1 > w_2 > \ldots > w_R$. No reward is received or penalty paid for rejecting calls. Our aim is to maximize the expected reward earned per unit time when the system is in equilibrium.

The case when holding times are exponentially distributed has been extensively studied. It was first shown by Miller[21] that for each type $r \in \mathcal{R}$, there exists a parameter t_r, such that if the number currently in the system is less than $C - t_r$, it is optimal (in terms of maximising the revenue earned per unit time in equilibrium), to accept type $\tilde{r}$ customers, $\tilde{r} \leq r$, and if the number in the system is greater than or equal to $C - t_r$ then type $\tilde{r}$ customers should be rejected for $\tilde{r} \geq r$. Furthermore, for the optimal policy, $0 = t_1 \leq t_2 \leq \ldots \leq t_R$. Such a policy is called a threshold policy or, in the telecommunications literature, a trunk reservation policy. Lippman[19] later gave a proof of this result using the technique of uniformization, which has been much used since then.

It is known that trunk reservation need not be optimal in more general cases. It is not in general optimal for any network consisting of more than a single link (Key[16] gives a numerical example of this in the case of a network with just two links). However, trunk reservation may be *asymptotically* optimal for networks with special structure (see Hunt and Laws[9] and MacPhee and Ziedins[20]). Hunt and Laws[10] have also shown that trunk reservation is asymptotically optimal for the single link when the assumptions on holding times and capacity requirements are weakened (although they still assume that holding times are exponentially distributed). The single link has also been examined by Ziedins[33], who considered the form of the optimal policy when *interarrival times* are distributed as a sum of exponentials. She found that a generalized trunk reservation policy, closely related to that of the examples given by us below, is optimal. Prior to that Nguyen[23] showed that a generalized trunk reservation policy is optimal for overflow traffic. A generalized trunk reservation policy is a threshold policy where the threshold may now depend on more than just the number of customers in the system – the threshold may vary with some other variable, such as, for instance, the number of calls in the first phase of their service. Other authors who have studied trunk reservation and threshold policies in telecommunication networks include Bean, Gibbens and Zachary[1], Kelly[14], Key[17] and Reiman[25]. Admission controls to queueing (rather than loss) systems, have been studied by, amongst others, Stidham[29], Johansen and Stidham[12], and Hordijk and Spieksma[8]. For an excellent overview of and introduction to the general area of loss networks see the review paper by Kelly[15].

The use of general phase-type or Coxian distributions has a long history (see, for instance, Cox[4]). The emphasis in most work has remained on single server facilities. Langen[18] used hyper-Erlang distributions for arrivals to single server queues and found optimal admission policies for $GI|M|C$ queues with batch admissions. Jo and Stidham[11] found policies for the control of service rates in an $M|G|1$ queue. In both cases results were first shown for phase-type distributions and then extended to general distributions. Stidham and Weber [30] used a different approach and established results for service rate control based on first step probabilities. However their method relies on passing through certain states and is not easily extended to multiserver systems with multi-dimensional state spaces.

There is also a relationship between tandem queues with exponential servers and single stations with phase service times, although most practical comparisons tend to be artificial. Examples of results for such systems are Nishimura[24], Ghoneim[6] and Ghoneim and Stidham[7]. The latter is particularly relevant as the structural properties of optimal policies for their model are almost identical to those we expect to find here. They consider two queues in series, and maximize a benefit function composed of rewards for entering customers minus holding costs for the two queues. The optimal policies possess various monotonicity properties – adding a customer to either queue makes it less likely that a new customer is accepted into either queue and moving a customer from the first queue to the second makes it more(less) likely that a new customer is accepted into the first(second) queue.

In this paper we will study the form of the policy that maximizes the expected return per unit time in equilibrium. We give examples showing that trunk reservation policies may do very nearly as well as the exactly optimal policy. It is known that the equilibrium distribution for an $M|G|C|C$ queue is insensitive to the form of the holding time distribution and depends on it only through its mean (see for example Burman, Lehoczky and Lim[3]). We show that this insensitivity still holds when simple trunk reservation controls (not of the generalized kind) are applied.

Section 2 outlines the Markov decision theory we will be using. In section 3 we discuss monotonicity properties and show that it is always optimal to accept the more valuable type 1 calls. Section 4 compares the optimal policy with trunk reservation and obtains an insensitivity result for the latter. We close with some final remarks in section 5.

2. Markov decision theory

We note first that since all calls have the same holding time distribution, the call types are indistinguishable from one another once they have entered the system. Let $\mathbf{x}(t) = \{x_1(t), \ldots, x_N(t)\}$ where $x_i(t)$ denotes the number of calls currently in phase i at time t. Then, since the arrivals are Poisson and phases are exponentially distributed, $\{\mathbf{x}(t), t \geq 0\}$ is a continuous-time Markov process with state space $S = \{\mathbf{x} : \sum x_i \leq C, x_i \in \mathbb{N} \cup \{0\}, 1 \leq i \leq I\}$.

We now review briefly the theory of Markov and semi-Markov decision processes as it applies to this problem. We follow the approach used in Tijms [31] and consider average cost processes only. The other common criterion used is that of discounted cost (for further details see Ross [26], [27], which also cover the average cost approach described here).

Markov decision theory can be characterized by a discrete time Markov chain in which decisions are made at the fixed epochs at which events may occur. We wish to choose a policy, $a \in \mathcal{A}$, the action space, for making decisions that is optimal in some way. If a is deterministic and does not vary with time then it is stationary. It can be shown (see, for instance, Tijms [31], §3.1 or Derman [5]), that as we are considering finite state spaces and any state is accessible from any other state we need only consider stationary policies.

The model analysed here is a loss network and the possible decisions that can be made are only whether to accept or reject an arriving call of type r. If we denote acceptance of a call by 1 and rejection by 0 then a stationary policy is a function $a_r : S \to \{0,1\}$, for each $r \in \mathcal{R}$.

In order to apply the theory of discrete-time Markov decision processes to this problem we use the uniformization technique first introduced by Lippman[19]. Let arrivals and holding time or phase completions occur at the same rates as before and introduce additional null transitions (events) which leave the system unchanged and occur at rate $\sum_{1\leq i\leq I}(C - x_i)\mu_i$ when the process is in state $\mathbf{x}$. Then the total rate at which transitions occur is now

$$\Lambda = \sum_{r\in\mathcal{R}} \lambda_r + C \sum_{1\leq i\leq I} \mu_i \tag{1}$$

whatever the state of the system. The underlying process is unchanged and a stationary optimal policy for the uniformized process is also optimal for the underlying process. Instead of studying the continuous-time uniformized process we can consider instead the discrete-time process with periods $1/\Lambda$ between jumps. In general the reward structure has to be altered (Serfozo [28]) – though we do not need to do so here – and only stationary policies can be used (Beutler and Ross [2]).

Denote the long term average reward per unit time under policy a by $g(a)$. Then if we denote the expected gain over the first n epochs starting in state $\mathbf{x}$ under policy a by $V_n(\mathbf{x}, a)$, $g(a)$ can be defined by

$$g(a) = \lim_{n\to\infty} \Lambda \frac{V_n(\mathbf{x}, a)}{n}. \tag{2}$$

So, if $\{\pi_{\mathbf{x}}(a)\}_{\mathbf{x}\in S}$ is the stationary distribution of the network under policy a, then

$$g(a) = \sum_{\mathbf{x}\in S} \pi_{\mathbf{x}}(a) \left(\sum_{r\in\mathcal{R}} \lambda_r a_r(\mathbf{x}) w_r \right). \tag{3}$$

We aim to maximise this return. This could be done by calculating the equilibrium distribution under each policy in $\mathcal{A}$. However this is impractical due to the size of the state space, S, and the number of policies, $|\mathcal{A}| \approx 2^{|S||\mathcal{R}|}$.

Two methods are generally used to find the optimal g. Both require the introduction of relative values for each state under a given policy, which we shall denote by $V(\mathbf{x})$ (strictly $V(\mathbf{x}, a)$ but we shall take the dependence on a given policy as read). Let $\mathbf{e}_i$ denote a vector of length I with 1 as the ith element and 0 for all other elements, and $\mathbf{e}_{I+1} = \mathbf{0}$. Then (Tijms [31], Theorem 3.1.1, Ross [27], Proposition 2.5), if the w_r and λ_r are bounded, the average cost function, $g(a)$, and the relative values, $V(\mathbf{x})$, satisfy the following system of linear equations: for $\mathbf{x} \in S$,

$$\begin{aligned} g(a) &= \sum_{r\in\mathcal{R}} \lambda_r \left[a_r(\mathbf{x})(V(\mathbf{x}+\mathbf{e}_1) + w_r) + (1 - a_r(\mathbf{x}))V(\mathbf{x})\right] \\ &\quad + \sum_{i=1}^{I} x_i\mu_i V(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}) - \left(\sum_r \lambda_r + \sum_i x_i\mu_i \right) V(\mathbf{x}). \end{aligned} \tag{4}$$

Let $\mathbf{x}^*$ be an arbitrary state. Then solving equations (4), combined with $V(\mathbf{x}^*) = 0$, gives a unique solution for $g(a)$ and the $V(\mathbf{x})$'s.

The relative values are only determined up to a constant and so it is differences between them that are important. The difference $V(\mathbf{x}) - V(\mathbf{y})$ has a natural economic interpretation as the price we are willing to pay to be in state $\mathbf{x}$ rather than state $\mathbf{y}$. Alternatively it is the difference in what we would expect to earn in the future if the process started in state $\mathbf{x}$ rather than state $\mathbf{y}$. Thus if we have $V_n(\mathbf{x}, a)$ as in equation (2) and n large, $V_n(\mathbf{x}, a) \approx \frac{ng(a)}{\Lambda} + V(\mathbf{x})$.

One method for calculating the optimal value of $g(a)$, g^*, and finding the optimal policy, is policy-iteration, which proceeds by fixing a policy and solving the Howard equations (4). Using the values of $V(\mathbf{x})$ given by this solution a new policy can be found by maximising the Howard equations over all $a \in \mathcal{A}$. Note that in the updated policy $a_r(\mathbf{x}) = 1$ if and only if

$$V(\mathbf{x} + \mathbf{e}_1) + w_r \ \geq \ V(\mathbf{x}). \tag{5}$$

In other words the call is accepted only if the decrease in relative value between states is less than the reward gained by accepting the call. If the policy is the same as before then we have found the optimal policy (see for instance Tijms [31], §3.2); otherwise the procedure is repeated for the new policy. It can be shown that this converges to the optimal policy.

A second method is value iteration (also known as the method of successive approximations). This proceeds by iteratively solving for the relative values while improving the policy at the same time. We start with an arbitrarily chosen initial function, $V_0(\mathbf{x})$, which is usually set equal to zero, and recursively calculate the following equations:

$$\begin{aligned} V_n(\mathbf{x}) \ = \ & \left\{ \sum_{r \in \mathcal{R}} \lambda_r V_{n-1}(\mathbf{x}, r) + \sum_{i=1}^{I} x_i \mu_i V_{n-1}(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1}) \right. \\ & \left. + \sum_{i=1}^{I} (C - x_i) \mu_i V_{n-1}(\mathbf{x}) \right\} / \Lambda \qquad \mathbf{x} \in S \end{aligned} \tag{6}$$

where

$$V_n(\mathbf{x}, r) = \begin{cases} \max\left(V_n(\mathbf{x} + \mathbf{e}_1) + w_r, V_n(\mathbf{x})\right) & \text{if } \sum x_i < C \\ V_n(\mathbf{x}) & \text{otherwise.} \end{cases}$$

It is shown in Tijms [31], §3.4 that if g_n is the return under the policy that maximises the right hand side of equations (6) at time n then $\Lambda m_n \leq g_n \leq g^* \leq \Lambda M_n$, where $m_n = \min_{\mathbf{x}}(V_n(\mathbf{x}) - V_{n-1}(\mathbf{x}))$ and $M_n = \max_{\mathbf{x}}(V_n(\mathbf{x}) - V_{n-1}(\mathbf{x}))$. Furthermore, m_n (M_n) is monotonically increasing (decreasing) in n. For aperiodic Markov chains (which we have here since uniformization implies that $p_{\mathbf{xx}} > 0$) these converge as indicated to g^*/Λ.

Generally the rate of convergence in value iteration is geometrically fast. For numerical calculations this can be speeded up by using a relaxation factor, although for theoretical results equations (6) are used directly. A common method for proving that a class of policy is optimal is to assume V_0 has a specific structure and to show inductively that this is preserved by the iterative procedure.

3. Structure of optimal policies and monotonicity

We wish to find the optimal control policies for our model. We might expect these to possess a certain structure. For example, we might expect that the busier the link, the less likely that calls of a given type would be accepted. We might also expect that the most valuable calls, that is, calls of type 1, should always be accepted as long as capacity is available to carry them.

To make this precise we first define what we mean by one state being busier than another. We define a partial ordering, $\prec$, (cf. Topkis [32]) on the state space, S. For this it is convenient to define a shift operator, T_i, such that if $\mathbf{x} \in S$,

$$T_i\mathbf{x} = \begin{cases} \mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i+1} & \text{if } 1 \leq i \leq I-1 \\ \mathbf{x} - \mathbf{e}_I & \text{if } i = I. \end{cases}$$

Obviously we preclude use of T_i on $\mathbf{x}$ if $x_i = 0$. We define

$$\prod_{n_j : n_j \in \mathcal{R}} T_{n_j} = \ldots T_{n_k} \ldots T_{n_1}$$

in the obvious way. Note that the ordering is important. For example if $x_j = 0$ and $x_{j-1} > 0$ then $T_j T_{j-1}\mathbf{x}$ is possible whereas $T_{j-1}T_j\mathbf{x}$ is not. However, if $x_j > 0$ and $x_{j-1} > 0$ then both give the same result. Then we can say that $\mathbf{x} \prec \mathbf{y}$ if there exist $\{n_j\}$ such that

$$\prod_{n_j : n_j \in \mathcal{R}} T_{n_j}\mathbf{y} = \mathbf{x}.$$

If $\mathbf{x} \prec \mathbf{y}$ then $\mathbf{x}$ is busier than $\mathbf{y}$. Thus we would expect that if $\mathbf{x} \prec \mathbf{y}$ then accepting a call of type r in state $\mathbf{y}$ implies that it is also accepted in state $\mathbf{x}$. Consideration of equation (5) leads us to propose that the optimal policy has the following properties. Let V^* denote the relative value function under the optimal policy a^*.

THEOREM 1 *Consider the single link model with R independent Poisson arrival streams having rate λ_r, $r \in R$. Let the call holding time distribution have I independent phases, the i^{th} phase being exponentially distributed with mean $1/\mu_i$. Calls of type r are worth w_r with $w_1 > w_2 > \ldots > w_R$. Let V^* denote the relative value function under the optimal policy a^*. Then for all $\mathbf{x}$ such that $\mathbf{x} + \mathbf{e}_1 \in S$,*

$$V^*(\mathbf{x}) - V^*(\mathbf{x} + \mathbf{e}_1) \leq w_1$$

that is, type 1 calls are always accepted if there is free capacity available.

Proof This follows immediately from the observation that the system cannot do better than accept a type 1 call when it is offered, since if it is rejected, the free capacity will either be used by a later call of type 1 (a delayed reward) or of some other type (both a reduced a and delayed reward) or, conceivably, not used at all. A formal proof using a coupling argument is given in [22]. ♠

Conjecture 1 *(a) For all* $\mathbf{x}$ *such that* $\mathbf{x}+2\mathbf{e}_1 \in S$,

$$V^*(\mathbf{x}) - V^*(\mathbf{x}+\mathbf{e}_1) \le V^*(\mathbf{x}+\mathbf{e}_1) - V^*(\mathbf{x}+2\mathbf{e}_1)$$

implying that if a call of type r *is accepted in state* $\mathbf{x}+\mathbf{e}_1$ *it is also accepted in state* $\mathbf{x}$.

(b) For all $\mathbf{x}$ *such that* $\mathbf{x}+\mathbf{e}_1 \in S$ *and* $x_i > 0$,

$$V^*(T_i\mathbf{x}) - V^*(T_i\mathbf{x}+\mathbf{e}_1) \le V^*(\mathbf{x}) - V^*(\mathbf{x}+\mathbf{e}_1)$$

implying that if a call of type r *is accepted in state* $\mathbf{x}$ *it is also accepted in state* $T_i\mathbf{x}$.

Conjecture 1(a) is that V^* is concave non-increasing in x_1. Conjecture 1(b) gives the value function a related regular structure called submodularity. For a further discussion of this see Topkis [32]. Repeated use of Conjecture 1(b) also gives the property that if $\mathbf{x} \prec \mathbf{y}$ then acceptance of a type r call in state $\mathbf{y}$ implies that it is accepted in state $\mathbf{x}$ as well. In particular, repeated application of Conjecture 1(b) implies Conjecture 1(a). Note that a trunk reservation policy *may* satisfy Conjecture 1.

We note that (5) trivially demonstrates that the optimal policy has the obvious property that if calls of type r are accepted in state $\mathbf{x}$ then calls of type $r' < r$ are also accepted in state $\mathbf{x}$, since $w_{r'} \ge w_r$.

We have been unable to prove Conjecture 1. Networks where proofs of expected policy structures cannot be produced are known (see Ghoneim [6]) and we do not consider this problem further here. We note that no numerical examples have been found which contradict this conjecture.

4. Trunk reservation and numerical examples

Although trunk reservation is not in general the optimal control policy for a single link loss network with phase holding times, it is known to be fairly robust to parameter changes, such as increases in the arrival rates (Key [16] gives a good example of this). In practical circumstances the distributions of the inter-arrival times and service times and their associated parameters will not be known exactly and a strategy that is fairly insensitive to deviations from assumptions is obviously desirable.

In this section we begin by showing that the optimal choice of trunk reservation parameter is insensitive to the call holding time distribution except through its mean. We then use numerical examples to compare the best trunk reservation policy with the optimal strategies when the holding times have phase distributions. We give examples showing that the the optimal trunk reservation policy gives a return that is very close to optimal for phase holding times with a reasonably wide range of parameters. We mostly consider examples with two arrival streams and two stage phase holding times.

THEOREM 2 *Consider a single-link loss system with independent Poisson arrival streams at rate λ_r, $r \in \mathcal{R}$. Let call holding times have a general distribution with finite mean and variance. Then the optimal stationary policy based solely on the number of calls in the system is trunk reservation. Moreover the optimal trunk reservation parameters are the same as those for exponential holding times with the same mean, that is, the optimal control policy is insensitive to the call holding time distribution except through its mean.*

Proof Let $n(t)$ be the total number of calls in the system at time t, regardless of their phase, i.e. $n(t) = \sum_i x_i(t)$. We add to the description of the state "flip-flop" variables, which can take the values 0 and 1. Each arrival stream is assigned a "flip-flop" variable, and the r^{th} such variable changes state whenever a call of type r arrives, but is not admitted into the system. Then with this modification to the state description we have a symmetric queue (Kelly[13]), and the equilibrium distribution for such a queue depends on the service time (call holding time) distribution only through its mean (Kelly[13], Theorem 3.10). Hence for any stationary admission policy its equilibrium distribution is the same as that of the same system with exponentially distributed holding times with the same mean. Now, if holding times are exponentially distributed, then the optimal policy is trunk reservation (Miller[21]), and so it will be the optimal policy for a general holding time distribution as well. Moreover, the optimal trunk reservation parameter will be the same as for exponentially distributed service times with the same mean. ♠

THEOREM 3 *Consider the single link model with R independent Poisson arrival streams having rate λ_r, $r \in R$. Let the call holding time distribution have I independent phases, the i^{th} phase being exponentially distributed with mean $1/\mu_i$. Calls of type r are worth w_r with $w_1 > w_2 > \ldots > w_R$. Suppose that there is a trunk reservation parameter t_r against type r calls, where since we have $w_1 > w_2 > \ldots > w_R$, we will have $0 = t_1 \le t_2 \le \ldots \le t_R$. Define $t_{R+1} = C$. Let x_i denote the number of calls in the i^{th} service phase. Then the equilibrium distribution is given by*

$$\pi(\mathbf{x}) = G \frac{\left(\sum_{l=1}^{j} \lambda_l\right)^{\sum_i x_i - C + t_{j+1}} \left(\prod_{k=j+1}^{R} \left(\sum_{l=1}^{k} \lambda_l\right)^{t_{k+1}-t_k}\right) \prod_{i=1}^{I} \left(\frac{1}{\mu_i}\right)^{x_i}}{\prod_{i=1}^{I} x_i!} \tag{7}$$

for $C - t_{j+1} < \sum_i x_i \le C - t_j$, $0 \le j \le R+1$, where $G = \pi(\mathbf{0})$ is a normalizing constant.

Proof. This can be found directly by considering the full balance equations. Alternatively, note that the $\mathbf{x}$ process is equivalent to a closed migration process with C individuals in it, and $I+1$ colonies, labelled $0, 1, \ldots, I$, where being in the i^{th} colony corresponds to being in the i^{th} service phase, and the 0^{th} colony contains a reservoir of individuals who are not being serviced (they will form the arrival process). Following the notation of Kelly ([13], Theorem 2.3), the parameters for the process are $\lambda_{01} = \lambda_1$, $\lambda_{i,i+1} = \mu_i$, $1 \le i < I$ and $\lambda_{I0} = \mu_I$ with $\phi_i(x_i) = x_i$, $1 \le i \le I$. The multipliers $\phi_0(\cdot)$ are a special case – they are given by $\phi_0(x_0) =$

$\sum_{j=1}^{r} \lambda_j/\lambda_1$ if $t_r < x_0 \leq t_{r+1}$, $1 \leq r \leq R$ with $\phi_0(0) = 0$, to allow for the changes in arrival rates with increasing occupancy. The result above then follows from [13], after some algebraic manipulation. ♠

An example of the distribution given by equation 7 is illustrated in Figure 1.

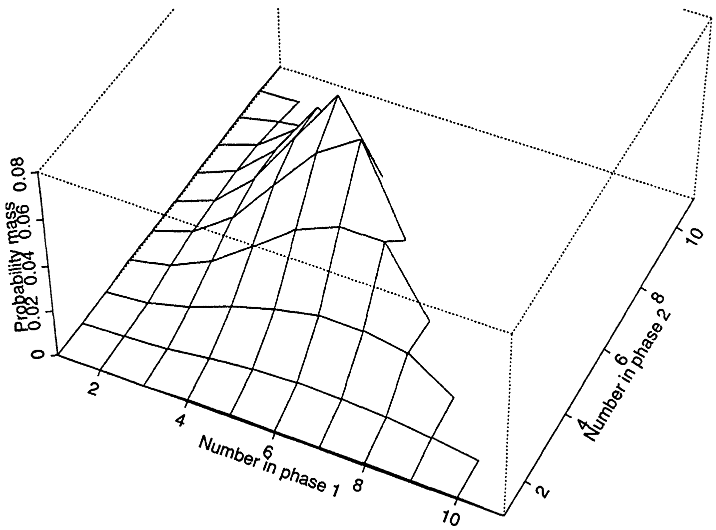

Figure 1. Perspective plot of the equilibrium distribution for Example 1.1 in Table 1.

We now compare the returns under different policies. Initially we consider some systems in which calls spend an equal time in each phase, with parameters as given in Table 1. We use three basic examples – one is a very overloaded network, one is underloaded and the third has a total load between the other two. The range of total arrival rates varies from .8 of capacity to over twice capacity. We also consider three different capacities, $C = 10, 40$ and 100. Note that for each of these examples the chosen parameters give an expected total holding time of 1 for each call.

First we give an example of an optimal policy. Table 2 shows the optimal acceptance/rejection policy for Example 1.1. Note that since the optimal policy always accepts type one calls when $x_1 + x_2 < C$ we need only consider the policy for type 2 calls.

Table 3 presents the returns under different policies for the examples given in Table 1. The calculation of the optimal return in each example was done using value iteration, with the convergence condition $M_n - m_n \leq \epsilon m_n$ used with $\epsilon = 10^{-4}$ (cf. Tijms [31]). The "T.R." return is that for phase holding times with the best choice of trunk reservation parameter. It can be seen that the gain by using the optimal policy instead of simple trunk reservation in these examples is almost negligible (at most 0.2%). The effect of network capacity on this gain can be seen to be small as

Table 1. Parameters used in examples.

Example	C	λ_1	λ_2	μ_1	μ_2	w_1	w_2
1.1	10	12	12	2	2	2	1
1.2	10	6	6	2	2	2	1
1.3	10	4	4	2	2	2	1
2.1	40	48	48	2	2	2	1
2.2	40	24	24	2	2	2	1
2.3	40	16	16	2	2	2	1
3.1	100	120	120	2	2	2	1
3.2	100	60	60	2	2	2	1
3.3	100	40	40	2	2	2	1

Table 2. Optimal values of $a_2(x_1, x_2)$ for Example 1.1.

		x_2										
		0	1	2	3	4	5	6	7	8	9	10
	0	1	1	1	1	1	1	1	0	0	0	0
	1	1	1	1	1	1	0	0	0	0	0	
	2	1	1	1	1	0	0	0	0	0		
	3	1	1	0	0	0	0	0	0			
	4	1	0	0	0	0	0	0				
x_1	5	0	0	0	0	0	0					
	6	0	0	0	0	0						
	7	0	0	0	0							
	8	0	0	0								
	9	0	0									
	10	0										

well, with perhaps a slight tendency for the relative difference to be smaller for the larger networks.

Table 3. Returns for the examples in Table 1.

Example	optimal T.R. parameter	optimal return	T.R. return
1.1	4	16.775	16.771
1.2	1	13.046	13.022
1.3	1	10.584	10.575
2.1	13	74.464	74.462
2.2	3	58.681	58.635
2.3	1	46.804	46.803
3.1	26	192.90	192.90
3.2	4	152.74	152.69
3.3	1	119.56	119.56

We now look at the effect of varying the relative time spent in each phase. We consider each of examples 1.1 to 1.3, but modify them by varying the mean time in each phase in such a way that the expected total holding time remains 1. In particular this means that the optimal trunk reservation parameter for each example is the same as before. The optimal returns for our modified examples are reported in Table 4. As can be seen there is again very little difference between the returns under the optimal and trunk reservation policies (the greatest relative difference being no more than 0.4%). As would be expected, these differences tend to 0 as μ_1 or μ_2 tends to 1.

Table 4. Optimal returns for modified examples 1.1 to 1.3.

Ex.	$\mu_1 = \frac{32}{31}$ $\mu_2 = 32$	$\mu_1 = \frac{16}{15}$ $\mu_2 = 16$	$\mu_1 = \frac{8}{7}$ $\mu_2 = 8$	$\mu_1 = \frac{4}{3}$ $\mu_2 = 4$	$\mu_1 = 4$ $\mu_2 = \frac{4}{3}$	$\mu_1 = 8$ $\mu_2 = \frac{8}{7}$	T.R.
1.1	16.774	16.777	16.779	16.778	16.772	16.771	16.771
1.2	13.042	13.071	13.071	13.064	13.028	13.023	13.022
1.3	10.600	10.602	10.601	10.595	10.576	10.575	10.575

In the next group of examples we examine the effect of adding additional phases to the service time distribution. We confine ourselves to considering cases with $C = 10$, since at larger capacities the relative difference between the returns under the optimal and best trunk reservation policies was less. In this case we again consider the examples 1.1 to 1.3, but modify the the service time distribution to have three phases with $\mu_1 = \mu_2 = \mu_3 = 3$, so that the mean of the total holding time remains unchanged at 1. The returns for the optimal policy are given in Table 5, with the T.R. return taken from Table 2. A comparison with Table 2 shows that, although the relative improvement to be obtained by using the optimal policy has increased very slightly, it is still negligible.

Table 5. Returns for examples with three phases.

Modified example	optimal return	T.R. return
1.1	16.778	16.771
1.2	13.059	13.022
1.3	10.591	10.575

Finally, we examine the effect of adding a third call type, again looking only at examples with $C = 10$. For these examples we have two phases with $\mu_1 = \mu_2 = 2$, as before. The rewards are now $w_1 = 2.0$, $w_2 = 1.0$ and $w_3 = 0.5$, with $\lambda_1 = \lambda_2 = \lambda_3$. The results for three values of λ_i are displayed in Table 6. Again, we see that adding additional call types does not substantially increase the relative improvement to be obtained by using the optimal policy - it remains negligible, with the greatest improvement here being .16%.

Table 6. Returns for examples with three types of call.

C	λ_i	optimal T.R. parameters t_2	t_3	optimal return	T.R. return
10	8	2	10	14.642	14.630
10	4	1	3	10.684	10.667
10	2.5	0	1	7.988	7.978

Two factors appear to be contributing to these observations that the best trunk reservation policy does nearly as well as the optimal policy. The first is that in those cases where simple trunk reservation *is* optimal (e.g. when holding times are exponentially distributed) it has been observed that a choice of trunk reservation parameter close to the optimal one generally gives performance not far from optimal – that is, the reward function is not sharply peaked around the optimal value of the trunk reservation parameter. The second factor is that it seems that the equilibrium distribution for the processes considered here will, in general, be unimodal (see, for example, Figure 1). Thus, within the region of high probability, a single choice of trunk reservation parameter will do rather well. And states outside this region, where the best choice of trunk reservation parameter may be very different, will have very low probability. For additional discussion see Hunt and Laws[10].

5. Concluding Remarks

We have seen that the revenue benefits of using the optimal policy rather than the best trunk reservation policy can be small. There are also additional overheads in implementing the optimal policy – it first needs to be calculated, and when it is in use there will be costs attached to the additional knowledge required about the state of the system, and to the greater complexity of the admission rule. Thus the simple trunk reservation policy, even for relatively small systems, may give very good performance, as well as being simple to implement and relatively robust.

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