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Classroom Note

A NOTE ON CALCULATING STEADY STATE RESULTS FOR AN M/M/k QUEUING SYSTEM WHEN THE RATIO OF THE ARRIVAL RATE TO THE SERVICE RATE IS LARGE

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Abstract. The formulas for calculating steady state results for the M/M/k queuing system are well known. Computational difficulties, however, can arise in using these formulas in situations in which the ratio of the arrival rate to the service rate is high. In this paper we consider a recursive approach for calculating steady state results which avoid such computational difficulties. As an additional benefit, such formulas are far easier to program into a spreadsheet package than the traditional formulas presented in most management science texts.

Keywords: Queue, Spreadsheet, Recursion, Steady State

The steady state formulas for an M/M/k queuing system with arrival rate λ and service rate μ are well known and presented in most introductory management science texts (see, for example, Lawrence and Pasternack [1998]). Since these formulas are based on $P_0(k)$, the probability of 0 customers in a system in which there are k servers, difficulties can arise when $a = \frac{\lambda}{\mu}$, the ratio of the arrival rate to the service rate, is large. This is because for large values of a, $P_0(k)$ will be effectively 0, resulting in values of 0 for the steady state service measures. Such situations can arise in airline reservations systems, computer support operations, credit card bureaus, etc.

The formula for $P_0(k)$ is as follows:

$$P_0(k) = \frac{1}{\sum_{i=0}^{k-1} \frac{a^i}{i!} + \frac{a^k}{(k-1)!(k-a)}} \qquad \text{where } a = \frac{\lambda}{\mu} < k \tag{1}$$

The average number of customers waiting for service, $L_q(k)$, is given by:

$$L_q(k) = P_0(k) \frac{a^{k+1}}{(k-1)!(k-a)^2} \tag{2}$$

while the probability of n customers in the system, $P_n(k)$, is given by:

$$P_n(k) = P_0(k)\frac{a^n}{n!} \quad \text{for } n \leq k \tag{3}$$

or

$$P_n(k) = P_0(k)\frac{a^n}{k!k^{n-k}} \quad \text{for } n > k \tag{4}$$

The remaining steady state results can be derived from $L_q(k)$ using Little's formulas or by using simple manipulation. These results are as follows:

The average number of customers in the system: $L = L_q(k) + a$ (5)

The average time a customer spends in the system: $W = \frac{L}{\lambda}$ (6)

The average time a customer spends in the queue: $W_q(k) = \frac{L_q(k)}{\lambda}$ (7)

The probability an arriving customer waits for service: $P_W(k) = \frac{L_q(k)(k-a)}{a}$ (8)

As can be seen from equation (1), for large values of a ($k > a$), $P_0(k)$ will have a value close to 0. In fact, it is possible for the denominator in equation (1) to become so large that either a computational overflow occurs or $P_0(k)$ is truncated to 0. In such cases, the values for the remaining steady state service measures will be in error.

For example, if $a \geq 130$, using the spreadsheet program *Excel* to calculate the value of $L_q(k)$ from equation (2) gives a cell value of #NUM! (an error message) whenever $k > 146$. If $a = 140$, *Excel* gives a value of #NUM! for $L_q(k)$ when $k = 144$ (instead of the correct value of 22.58674). For values of $a \geq 145$, *Excel* cannot calculate $L_q(k)$ or $P_n(k)$ for any value of k. The situation is even worse in some of the dedicated management science software packages. For example, *WINQSB* gives a 0 value for $L_q(k)$ whenever $a > 88$.

One possible way to remedy this situation is to use a recursive approach for calculating $P_0(k)$, $P_n(k)$, and $L_q(k)$. To do this we define:

$$V(k) = \sum_{i=0}^{k-1} \frac{k!}{i!} a^{i-k-1} \tag{9}$$

Hence, $V(1) = \frac{1}{a}$ and

$$V(k+1) = \left(\frac{k+1}{a}\right)(V(k)+1) \tag{10}$$

¿From equations (1) and (10) we see that:

$$P_0(k) = \frac{\frac{k!}{a^k}}{V(k) + \frac{k}{k-a}} \tag{11}$$

while from equations (2) and (10) we see that:

$$L_q(k) = \frac{ka}{(k-a)^2\left(V(k) + \dfrac{k}{k-a}\right)} \tag{12}$$

The formulas for $P_n(k)$ can be easily shown to be:

$$P_n(k) = \frac{\dfrac{k!}{n!}}{a^{k-n}\left(V(k) + \dfrac{k}{k-a}\right)} \quad \text{for } n \leq k \tag{13}$$

and

$$P_n(k) = \frac{\left(\dfrac{a}{k}\right)^{n-k}}{V(k) + \dfrac{k}{k-a}} \quad \text{for } n > k \tag{14}$$

The above formulas will work for any value of a (although for values of $a \geq k$ the values calculated by the formulas are meaningless) and are easy to program in a spreadsheet. For example, consider the case where $a = 1000$. If $k = 1001$, we see from equation (1) that $P_0(1001) \approx e^{-1000} \approx 0$. Using a spreadsheet program to calculate equation (9) recursively, gives a value of $V(1001) = 40.34352$ and from equation (12) we obtain the value of 961.2582 for $L_q(1001)$. In the case where $a = 145$, $P_{140}(146)$ is calculated to be .005801 using equation (13) while $P_{147}(146)$ is calculated to be .011036 using equation (14).

Reference

1. J. Lawrence, Jr. and B. A. Pasternack. Applied Management Science, A Computer-Integrated Approach for Decision Making. John Wiley & Sons, New York, 1998.