Δ-MATROID AND JUMP SYSTEM

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The concept of Δ-matroid is a nontrivial, proper generalization of the concept of matroid and has been further generalized to the concept of jump system. In this paper, we show that jump systems are, in some sense, equivalent to Δ-matroids. Using this equivalence and the Δ-matroid theory, we give simple proofs and extensions of many of the results on jump systems.

1. Introduction

In [7, 10], the concept of pseudomatroid was developed as a proper generalization of the concept of matroid. The same concept was independently developed as Δ-matroid in [4, 5]. Throughout the paper, we use the more popular name Δ-matroid for this structure.

In [6], the concept of Δ-matroid was further generalized to jump system. Further interesting results on jump system are reported in [1, 3, 12, 14, 15].

In this paper, we show that jump systems are, in some sense, equivalent to Δ-matroids. Using this equivalence and the Δ-matroid theory, we give simple proofs and extensions of many of the results on jump systems in [3, 6, 12].

In Section 2, we introduce notations and basic definitions. In Section 3, we give known and some new results on Δ-matroid. In Section 4, we prove equivalence between jump systems and Δ-matroids. We use this equivalence to give simple proofs of some of the known results on jump systems in Section 5.

2. Notations, definitions, and some basic results

Standard notations as in [7, 16] are used throughout. We also assume basic knowledge of matroid theory. In particular, the following notations and definitions are used extensively.

For any finite set E, we denote by 2^E the collection of all subsets of E. Let $3^E \equiv \{(A,B) : A,B \subseteq E, A \cap B = \emptyset\}$. For any $(A,B),(C,D) \in 3^E$, we write $(A,B) \sqsubseteq (C,D)$ if $A \subseteq C$ and $B \subseteq D$. For any $(A,B) \in 3^E$, $P_{A,B} = \{(X,Y) : (X,Y) \in 3^E, (X,Y) \sqsubseteq (A,B)\}$.

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We use the reduced union, $\sqcup$, and reduced intersection, $\sqcap$, of elements of 3^E defined as follows.

For (A_1,B_1) and $(A_2,B_2) \in 3^E$,

$$\begin{gathered}(A_1,B_1) \sqcup (A_2,B_2) = \big((A_1 \cup A_2) - (B_1 \cup B_2), (B_1 \cup B_2) - (A_1 \cup A_2)\big), \\ (A_1,B_1) \sqcap (A_2,B_2) = (A_1 \cap A_2, B_1 \cap B_2).\end{gathered} \tag{2.1}$$

A subset Ω of 3^E is called a signed-ring family if Ω is closed under the operations of $\sqcup$ and $\sqcap$.

A function $f : \Omega \to \mathbb{R}$ over a signed-ring family $\Omega \subseteq 3^E$ is said to be bisubmodular (bisupermodular) if it satisfies $f(X_1,Y_1) + f(X_2,Y_2) \geq (\leq) f((X_1,Y_1) \sqcap (X_2,Y_2)) + f((X_1,Y_1) \sqcup (X_2,Y_2))$ for any $(X_i,Y_i) \in \Omega$, $i = 1,2$.

A discrete system is a pair $(E,\mathfrak{I})$, where E is a finite set and $\emptyset \neq \mathfrak{I} \subseteq 2^E$.

Definition 2.1 [4, 10]. A discrete system $(E,\mathfrak{I})$ is a Δ-matroid (or pseudomatroid) if the set $\mathfrak{I}$ satisfies the following symmetric exchange axiom.

For any $A,B \in \mathfrak{I}$ and any $x \in A\Delta B$, either $A\Delta\{x\} \in \mathfrak{I}$ or there exists $y \in A\Delta B$ such that $A\Delta\{x,y\} \in \mathfrak{I}$.

The elements of $\mathfrak{I}$ are called independent sets of the Δ-matroid.

For any finite set E and any $X \subseteq E$, the incidence vector of X is a $(0,1)$-vector $x \in \mathbb{Z}^{|E|}$ such that $x_i = 1$ if and only if $i \in X$. We call the set X the support of the vector x. Also, for any $(X,Y) \in 3^E$, the incidence vector of (X,Y) is a $(0,\pm 1)$-vector $z \in \mathbb{Z}^{|E|}$ such that $z_i = 1$ (-1) if and only if $i \in X$ $(i \in Y)$. We call (X,Y) the support of z. For any $A \subseteq E$ and $x \in \mathbb{R}^E$, $x[A] = \sum\{x_i : i \in A\}$. For any $(A,B) \in 3^E$ and $x \in \mathbb{R}^E$, $x[A,B] = x[A] - x[B]$.

Definition 2.2 [2, 9]. Let $f : 3^E \to \mathbb{Z}$ be a nondecreasing, bisubmodular function. Then,

$$\wp \equiv \{x : x \in \mathbb{Z}^n;\ x[A,B] \leq f(A,B)\ \forall (A,B) \in 3^E\} \tag{2.2}$$

is a bisubmodular system.

For any $x,y \in \mathbb{R}^n$, we use the ℓ_1-norm $\|x\| = \sum_{i=1}^n |x_i|$ and the ℓ_1-distance $d(x,y) = \|x-y\|$. For any $S_1,S_2 \subseteq \mathbb{R}^n$, $d(S_1,S_2) = \min\{d(x,y) : x \in S_1, y \in S_2\}$. Define a function $f : S_1 \to \mathbb{R}$ as follows: for any $x \in S_1$, $f(x) = \min\{d(x,y) : y \in S_2\}$. Then it is easy to see that when S_1 and S_2 are convex sets, f is a separable, convex function.

A $(0,\pm 1)$-vector with a unique nonzero element is called a step. Occasionally, we denote a step by χ^i, implying that its ith element is the nonzero element. For any $x,y \in \mathbb{Z}^n$, a step from x to y is a step s such that $\|x+s-y\| = \|x-y\| - 1$. Let $\mathrm{St}(x,y)$ denote the set of all the steps from x to y.

Definition 2.3 [6]. A jump system $\mathbf{J} \subseteq \mathbb{Z}^n$ is a nonempty set satisfying the following 2-step axiom.

For any $x, y \in \mathbf{J}$ and any $s \in \mathrm{St}(x, y)$ with $x + s \notin \mathbf{J}$, there exists $t \in \mathrm{St}(x + s, y)$ such that $x + s + t \in \mathbf{J}$.

It is easy to see that if $(E, \mathfrak{I})$ is a Δ-matroid, then the set of incidence vectors of elements of $\mathfrak{I}$ forms a jump system and if all the elements of a jump system $\mathbf{J} \subseteq \mathbb{Z}^n$ are $(0,1)$-vectors, then the set $\mathfrak{I}$ of supports of elements of $\mathbf{J}$ forms a Δ-matroid.

For sets $S_1 \in \mathbb{R}^n$ and $S_2 \in \mathbb{R}^m$, let $S_1 \otimes S_2 = \{(x, y) : x \in S_1 \text{ and } y \in S_2\}$. Let $a_i \leq b_i$ be integers for $i = 1, \ldots, n$. The set of integer points in $[a_1, b_1] \otimes \cdots \otimes [a_n, b_n]$, denoted by $\mathbf{B}$, is called a box. It is easy to see that every box is a jump system [12].

Theorem 2.4 [6]. *Every bisubmodular system is a jump system and for any jump system, $\mathbf{J} \subseteq \mathbb{Z}^n$, the set of integer points in the convex hull of $\mathbf{J}$ forms a bisubmodular system.*

For other interesting examples of jump system, the reader is referred to [6, 12].

Definition 2.5. For any $x \in \mathbb{R}^n$, let the components x_i of x be ordered as $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. For any $x, y \in \mathbb{R}^n$, if $\sum_{i=1}^{j} x_{[i]} \leq \sum_{i=1}^{j} y_{[i]}$ for all $j = 1, 2, \ldots, n$, then x is said to be weakly submajorized by y and denoted by $x \preceq_w y$. If $-x \preceq_w -y$, then x is said to weakly supermajorized by y and denoted by $x \preceq^w y$. For any subset S of $\mathbb{R}^n$, $x \in S$ is a least weakly submajorized (supermajorized) element of S if $x \preceq_w y$ (resp., $x \preceq^w y$) holds for all $y \in S$.

The following theorem characterizes least weakly sub- and supermajorized elements.

Theorem 2.6 [13]. *For any nonempty subset $S \subseteq \mathbb{R}^n$, x^* is a least weakly submajorized (supermajorized) element of S if and only if for any continuous, nonincreasing (resp., nondecreasing) concave function $f : \mathbb{R} \to \mathbb{R}$, x^* is an optimal solution of the following problem: maximize $\{\sum_{i=1}^{n} f(x_i) : x \in S\}$.*

3. Results on Δ-matroids

In this section, we present known and some new results on Δ-matroids. These will be used to obtain simple proofs of results on jump systems in Section 5.

Fact 1 [10]. If $(E, \mathfrak{I})$ is a Δ-matroid, then all the maximal (minimal) elements of $\mathfrak{I}$ have the same cardinality and the set of maximal (minimal) elements forms the set of basis of a matroid.

Fact 2 [5]. For any discrete system $(E, \mathfrak{I})$ and any $X \subseteq E$, let $\mathfrak{I}\Delta X \equiv \{Y \Delta X : Y \in \mathfrak{I}\}$. Then $(E, \mathfrak{I})$ is Δ-matroid if and only if $(E, \mathfrak{I}\Delta X)$ is a Δ-matroid.

For disjoint, finite sets E_1 and E_2, if $(E_1, \mathfrak{I}_1)$ and $(E_2, \mathfrak{I}_2)$ are Δ-matroids, then it is easy to see that $(E_1 \cup E_2, \mathfrak{I}_1 \otimes \mathfrak{I}_2)$ is a Δ-matroid, where $\mathfrak{I}_1 \otimes \mathfrak{I}_2 = \{X \cup Y : X \in \mathfrak{I}_1 \text{ and } Y \in \mathfrak{I}_2\}$.

Definition 3.1 [7, 10]. The rank function $r : 3^E \to \mathbb{Z}_+$ of a Δ-matroid $(E, \mathfrak{I})$ is defined as

$$r(A, B) = \max\{|X \cap A| - |X \cap B| : X \in \mathfrak{I}\} \quad \forall (A, B) \in 3^E. \tag{3.1}$$

THEOREM 3.2 [7, 10]. *A function $r : 3^E \to \mathbb{Z}_+$ is the rank function of a Δ-matroid if and only if it satisfies the following:*

(i) $r(\cdot,\cdot)$ *is a bisubmodular function;*
(ii) $r(\emptyset,\emptyset) = 0$;
(iii) $0 \le r(i,\emptyset) \le 1$ *for all* $i \in E$;
(iv) $\{(A,B),(C,D) \in 3^E; (A,D) \sqsubseteq (C,B)\} \Rightarrow r(A,B) \le r(C,D)$.

THEOREM 3.3 (polyhedral characterization of **Δ**-*matroid)* ([7, 10]). *For any function $f : 3^E \to \mathbb{Z}_+$, consider the polyhedron*

$$\wp = \{x : x[A,B] \le f(A,B)\ \forall (A,B) \in 3^E\}. \tag{3.2}$$

Then $\wp$ is a Δ-matroid polytope (i.e., it is the convex hull of incidence vectors of independent sets of a Δ-matroid on E) if and only if the function $f(\cdot,\cdot)$ satisfies conditions of Theorem 3.2 and in this case, $f(\cdot,\cdot)$ is the rank function of the corresponding Δ-matroid.

Consider the following linear optimization problem on a discrete system $(E,\mathfrak{I})$.
Given $c : E \to \mathbb{Z}$, find $X \in \mathfrak{I}$ such that

$$c[X] = \max\{c(Y) : Y \in \mathfrak{I}\}. \tag{3.3}$$

We will now present three greedy algorithms for problem (3.3), each of which produces an optimal solution if $(E,\mathfrak{I})$ is a Δ-matroid.

Generalized greedy algorithm (I) (GGA(I)) [4, 7, 8, 10].

Let the elements of $E = \{1,2,\ldots,n\}$ be ordered such that $|c(1)| \ge |c(2)| \ge \cdots \ge |c(n)|$.
Step 0. $X^0 = \emptyset$; $i = 1$.
Step 1. If $c(i) > 0$, then
    if there exists $Y \subseteq \{i+1,\ldots,n\}$ such that $X^{i-1} \cup \{i\} \cup Y \in \mathfrak{I}$,
    then $X^i = X^{i-1} \cup \{i\}$.
    Else, $X^i = X^{i-1}$.
    Go to Step 2.
  If $c(i) < 0$, then
    if there exists $Y \subseteq \{i+1,\ldots,n\}$ such that $X^{i-1} \cup Y \in \mathfrak{I}$, then $X^i = X^{i-1}$.
    Else, $X^i = X^{i-1} \cup \{i\}$.
    Go to Step 2.
  Else, choose any $Y \subseteq \{i,i+1,\ldots,n\}$, such that $X^{i-1} \cup Y \in \mathfrak{I}$.
    Let $X^* = X^{i-1} \cup Y$. Stop.
Step 2. If $i < n$, then $i = i+1$; go to Step 1.
  Else, $X^* = X^i$; stop.

THEOREM 3.4 [4, 7, 10]. *$(E,\mathfrak{I})$ is a Δ-matroid if and only if GGA(I) produces an optimal solution to problem (3.3) for all $c : E \to \mathbb{Z}$.*

Fact 3 [7, 10]. For a Δ-matroid $(E,\mathfrak{I})$ with rank function $r(\cdot,\cdot)$, and any $(A,B) \in 3^E$, let $\mathfrak{I}/(A,B) = \{X - (A \cup B) : X \in \mathfrak{I}, |X \cap A| - |X \cap B| = r(A,B)\}$. Then it follows easily from the validity of the greedy algorithm GGA(I) that $(E - (A \cup B), \mathfrak{I}/(A,B))$ is a Δ-matroid.

THEOREM 3.5. *Let $(E,\mathfrak{I})$ be a Δ-matroid with rank function $r(\cdot,\cdot)$. For any $(A,B) \in 3^E$, let $\mathfrak{I}_{A,B} = \{X : X \in \mathfrak{I}, |X \cap A| - |X \cap B| = r(A,B)\}$. Then $(E,\mathfrak{I}_{A,B})$ is a Δ-matroid.*

Proof. Let $\mathfrak{I}' = \{X \cap (A \cup B) : X \in \mathfrak{I}_{A,B}\}$. Then it is easy to see that $(E,\mathfrak{I}')$ is a Δ-matroid and that $\mathfrak{I}_{A,B} = \mathfrak{I}' \otimes (\mathfrak{I}/(A,B))$. □

From Theorems 3.4 and 3.5, we have the following corollary.

COROLLARY 1. *Every face of a Δ-matroid polytope is a Δ-matroid polytope.*

The algorithm GGA(I) requires an oracle to check the following.

Oracle 1. Given $X \subseteq \{1,2,\ldots,i\}$, is there a $Y \subseteq \{i+1,\ldots,n\}$ such that $X \cup Y \in \mathfrak{I}$?

It is easy to see that the complexity of algorithm GGA(I) is $O(n)$, where each call to Oracle 1 is counted as a single operation. In some instances, however, information about the discrete system is available not in the form of Oracle 1, but in the form of the following membership Oracle 2.

Oracle 2. Given $X \subseteq E$, does $X \in \mathfrak{I}$?

We give below algorithm GGA(III) which, starting with any $Y^0 \in \mathfrak{I}$, uses Oracle 2 to produce an optimal solution to problem (3.3). But before that we consider another algorithm GGA(II), which starts with any $T^0 \in \mathfrak{I}$, and is essentially algorithm GGA(I) applied to $(E, \mathfrak{I}\Delta T^0)$.

Generalized greedy algorithm (II) (GGA(II)).

Input: $T^0 \in \mathfrak{I}$. Elements of $E = \{1,2,\ldots,n\}$ are ordered such that
$|c(1)| \geq |c(2)| \geq \cdots \geq |c(n)|$.
Step 0. $i = 0$.
Step 1. $i = i+1$;
If $c[T^{i-1}\Delta\{i\}] > c[T^{i-1}]$, then
if there exists $X \subseteq \{i+1,\ldots,n\}$ such that $T^{i-1}\Delta\{i\}\Delta X \in \mathfrak{I}$, then
$T^i = T^{i-1}\Delta\{i\}$.
Else $T^i = T^{i-1}$.
Go to Step 2.
If $c[T^{i-1}\Delta\{i\}] < c[T^{i-1}]$, then
if there exists $X \subseteq \{i+1,\ldots,n\}$ such that $T^{i-1}\Delta X \in \mathfrak{I}$, then $T^i = T^{i-1}$.
Else $T^i = T^{i-1}\Delta\{i\}$.
Go to Step 2.
If $c[T^{i-1}\Delta\{i\}] = c[T^{i-1}]$, then
choose any $X \subseteq \{i,i+1,\ldots,n\}$ such that $T^{i-1}\Delta X \in \mathfrak{I}$.
Set $T^* = T^{i-1}\Delta X$ and stop.
Step 2. If $i < n$, then go to Step 1. Else, set $T^* = T^i$ and stop.

Since GGA(II) is GGA(I) applied to $(E, \mathfrak{I}\Delta T^0)$, the validity of this algorithm, when $(E, \mathfrak{I})$ is a Δ-matroid, follows from Fact 2 and Theorem 3.4.

Generalized greedy algorithm (III) (GGA(III)).

Input: $Y^0 \in \mathfrak{I}$. Elements of $E = \{1, 2, \dots n\}$ are ordered such that
$|c(1)| \geq |c(2)| \geq \cdots \geq |c(n)|$.
Step 0. $i = 0$.
Step 1. $i = i + 1$.
Let j_i be the smallest integer such that
$c[Y^{i-1}\Delta\{j_i\}] > c[Y^{i-1}]$ and
either (i) $Y^{i-1}\Delta\{j_i\} \in \mathfrak{I}$ or
(ii) $Y^{i-1}\Delta\{j_i\} \notin \mathfrak{I}$ and there exists $k_i \in E$ such that
$Y^{i-1}\Delta\{j_i, k_i\} \in \mathfrak{I}$.
If no such j_i exists, then stop with $Y^* = Y^{i-1}$.
Else, in case (i), let $Y^i = Y^{i-1}\Delta\{j_i\}$ and
in case (ii), $Y^i = Y^{i-1}\Delta\{j_i, k_i\}$.
If $i < n$, then repeat Step 1. Else, stop with $Y^* = Y^i$.

It is easy to see that for any $u < v$, (i) $j_u < j_v$ and (ii) $Y^u \cap \{1, 2, \dots, j_u\} = T^u \cap \{1, 2, \dots, j_u\}$, where T^u is as defined in algorithm GGA(II). This, together with Theorem 3.4, validity of algorithm GGA(II) and definition of Δ-matroid gives us the following theorem.

Theorem 3.6. *Algorithm GGA(III) terminates in no more than n iterations and can be implemented in $0(n^2)$ time. If $(E, \mathfrak{I})$ is a Δ-matroid, then it terminates with an optimal solution to problem (3.3).*

We also get the following corollary.

Corollary 2. *Let $(E, \mathfrak{I})$ be a Δ-matroid and let $X \in \mathfrak{I}$. Define $A_X = \{Y : Y \in \mathfrak{I}$; either $(Y\Delta X = \{i\}$ or $(Y\Delta X = \{i, j\}$ and $X\Delta\{i\} \notin \mathfrak{I})\}$. Then for any $c : E \to \mathbb{Z}$, X is an optimal solution to problem (3.3) if and only if for all $Y \in A_X$, $c[Y] \leq c[X]$. Furthermore, A_X is the minimum set satisfying this property.*

4. Jump system as Δ-matroid

Consider a jump system $\mathbf{J} \in \mathbb{Z}^n_+$. Let $\alpha_i = \max\{x_i : x \in \mathbf{J}\}$, $i = 1, \dots, n$, and $E = E_1 \cup E_2 \cup \cdots \cup E_n$ where $E_i = \{e_{i1}, e_{i2}, \dots, e_{i\alpha_i}\}$, $i = 1, 2, \dots, n$. For any $x \in \mathbf{J}$, let $\mathfrak{I}_x$ denote the family of subsets of E consisting of x_i elements of E_i, for each $i \in \{1, 2, \dots, n\}$. Define, $\mathfrak{I} = \bigcup\{\mathfrak{I}_x : x \in \mathbf{J}\}$.

For any $X, Y \in 2^E$, we call an element $e_{ij} \in E$ a step element from X to Y if and only if either $|X \cap E_i| > |Y \cap E_i|$ and $e_{ij} \in X - Y$ or $|X \cap E_i| < |Y \cap E_i|$ and $e_{ij} \in Y - X$. Let $\mathrm{ST}(X, Y)$ denote the set of all the step elements from X to Y. If $X \in \mathfrak{I}_x$ and $Y \in \mathfrak{I}_y$ for some $x, y \in \mathbf{J}$, then it is clear that any step $s \in \mathrm{St}(x, y)$ corresponds to a set of step elements from X to Y and conversely, for every step element $e_{ij} \in \mathrm{ST}(X, Y)$, there is a unique step $s \in \mathrm{St}(x, y)$.

Theorem 4.1. *For any jump system $\mathbf{J} \subseteq \mathbb{Z}^n_+$, the pair $(E, \mathfrak{I})$ defined above is a Δ-matroid.*

For proof of the above theorem, we refer the reader to [11]. We now prove the converse of the theorem. For this, we define a function $\zeta_{12} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ as $\zeta_{12}(x) = (x_1 + x_2, x_3, x_4, \dots, x_n)^t$ for any $x \in \mathbb{R}^n$. We can similarly define a function ζ_{ij} for any $1 \le i < j \le n$.

Theorem 4.2. *Let $\mathbf{J} \subseteq \mathbb{Z}^n$ be a jump system. Define $\mathbf{J}' = \{x' : x' = \zeta_{12}(x)$, for some $x \in \mathbf{J}\}$. Then $\mathbf{J}'$ is a jump system.*

Proof. If the result is not true, then amongst all the pairs of elements of $\mathbf{J}'$ violating the axiom of jump system, choose the pair x', y' and the corresponding $x, y \in \mathbf{J}$ (i.e., $x' = \zeta_{12}(x)$ and $y' = \zeta_{12}(y)$), such that $|x_1 - y_1| + |x_2 - y_2|$ is minimum. Let $s' \in \mathrm{St}(x', y')$ be such that $x' + s' \notin \mathbf{J}'$.

Case 1 ($s' = \chi^1$). In this case, there exists an $s \in \mathrm{St}(x, y)$ such that $s = \chi^1$ or χ^2 and $\zeta_{12}(x + s) = x' + s'$. Without loss of generality, let $s = \chi^1$. Then $x + s \notin \mathbf{J}$ and hence, there exists $t \in \mathrm{St}(x + s, y)$ such that $z = x + s + t \in \mathbf{J}$. If $\zeta_{12}(z) \neq x'$, then $\zeta_{12}(z) - (x' + s') \in \mathrm{St}(x' + s', y')$. If $\zeta_{12}(z) = x'$, then $|z_1 - y_1| + |z_2 - y_2| < |x_1 - y_1| + |x_2 - y_2|$. This contradicts the choice of x.

Case 2 ($s' = \chi^j$ for some $j > 1$). In this case, there exists an $s \in \mathrm{St}(x, y)$ such that $\zeta_{12}(x + s) = x' + s'$ and $x + s \notin \mathbf{J}$. Hence, there exists $t \in \mathrm{St}(x + s, y)$, such that $z = x + s + t \in \mathbf{J}$. If $t' = \zeta_{12}(z) - (x' + s') \in \mathrm{St}(x' + s', y')$, then we are done. Else, $|z_1 - y_1| + |z_2 - y_2| < |x_1 - y_1| + |x_2 - y_2|$ and $-t' \in \mathrm{St}(\zeta_{12}(z), y')$. Hence, there exists $u' \in \mathrm{St}(\zeta_{12}(z) - t', y') = \mathrm{St}(x' + s', y')$, such that $\zeta_{12}(z) - t' + u' = x' + s' + u' \in \mathbf{J}'$. This proves the result. □

As corollaries, we get the following results.

Corollary 3 [6]. *Sum of two jump systems $\mathbf{J_1}, \mathbf{J_2} \subseteq \mathbb{Z}^n$ is a jump system.*

Corollary 4. *Let $(E, \mathfrak{I})$ be a Δ-matroid and let $(E_1, E_2, \dots, E_k)$ be a partition of E. Define $\mathbf{J} \subseteq \mathbb{Z}_+^k$ as $\mathbf{J} = \{x = (x_1, x_2, \dots, x_k)^t$: there exists $X \in \mathfrak{I}$ such that $x_i = |X \cup E_i|$ for all $i = 1, 2, \dots, k\}$. Then $\mathbf{J}$ is a jump system.*

It is easy to see [6] that for any jump system $\mathbf{J} \in \mathbb{Z}^n$, and any vector $u \in \mathbb{Z}^n$, $\mathbf{J} + u = \{x + u : x \in \mathbf{J}\}$ is a jump system. Henceforth, when we refer to the Δ-matroid $(E, \mathfrak{I})$ corresponding to a jump system $\mathbf{J} \subseteq \mathbb{Z}^n$, we will mean the Δ-matroid constructed as in the beginning of this section from $\mathbf{J^1} = \mathbf{J} - a$, where $a_i = \min\{x_i : x \in \mathbf{J}\}$ for all $i = 1, 2, \dots, n$. Thus, $\alpha_i = \max\{x_i : x \in \mathbf{J}\} - a_i$ for all i.

In the next section, we will use the results on Δ-matroid and the equivalence between Δ-matroid and jump system proved here to give simple proofs of known results on jump systems.

5. Results on jump systems

We start with the results of Ando et al. [3], on maximization of a separable concave function over a jump system. Thus, for $\mathbf{J} \subseteq \mathbb{Z}^n$, consider the following problem:

$$\max\left\{f(x) = \sum_{i=1}^{n} f_i(x_i) : x \in \mathbf{J}\right\}. \tag{5.1}$$

where, each f_i is a concave function over reals.

We show below that problem (5.1) is a special case of problem (3.3) on the corresponding Δ-matroid.

Let $(E,\mathfrak{I})$ be the Δ-matroid corresponding to $\mathbf{J}$. For $i = 1,2,\dots,n$, define $g_i(0) = f_i(a_i)$ and $g_i(j) = f_i(a_i + j) - f_i(a_i + j - 1)$ for all $j \in \{1,2,\dots,\alpha_i\}$. (We recall that $a_i = \min\{x_i : x \in \mathbf{J}\}$ and $\alpha_i = \max\{x_i : x \in \mathbf{J}\} - a_i$.)

Let $g[0] = \sum_{i=1}^{n} g_i(0)$, and for any $X \subseteq E$, let $g[X] = \sum\{g_i(j) : e_{ij} \in X\}$. Consider the following linear optimization problem on $(E,\mathfrak{I})$:

$$\max\{g[X] : X \in \mathfrak{I}\}. \tag{5.2}$$

We call an element $X \subseteq E$ a leftmost element if and only if for any $i \in \{1,2,\dots,n\}$ and any $j > 1$, $e_{ij} \in X \Rightarrow e_{i(j-1)} \in X$. It is easy to see that for every $x \in \mathbf{J} - a$, the set $\mathfrak{I}_x$ contains a unique leftmost element.

The theorem below follows easily from the fact that g_i's are nonincreasing functions.

Theorem 5.1. *For any $x \in \mathbf{J}$, if $X \in \mathfrak{I}_{x-a}$ is a leftmost element, then*

$$f(x) = g[X] + g[0] = \max\{g[Y] : Y \in \mathfrak{I}_{x-a}\} + g[0]. \tag{5.3}$$

Hence, $x \in \mathbf{J}$ is an optimal solution to problem (5.1) if and only if the leftmost element of $\mathfrak{I}_{x-a}$ is an optimal solution to problem (5.2). Thus, an optimal solution to problem (5.2), and hence to problem (5.1), can be obtained in $0(\sum_{i=1}^{n}\alpha_i)$ time using any of the algorithms GGA(I), GGA(II), and GGA(III). In fact, it is easy to modify these algorithms to apply directly to problem (5.1). Such a modification of algorithm GGA(III) is considered in [3]. Their results are precisely Corollary 2 and Theorem 3.6 applied to problem (5.2), and hence to problem (5.1).

As a corollary, we also obtain a simple proof of the following result of Ando [1] and Tamir [15].

Corollary 5 [1, 15]. *Every jump system $\mathbf{J} \subseteq \mathbb{Z}^n$ has a least weakly submajorized element and a least weakly supermajorized element.*

Proof. The solution to problem (5.2), and hence to problem (5.1), produced by Algorithm GGA(I) depends on the objective function coefficients only through the ordering imposed on the elements of E which are arranged in nonincreasing order of absolute values of objective function coefficients. It is easy to see that if for all $i = 1,2,\dots,n$, $f_i = f$ for some nonincreasing (nondecreasing) concave function f, then there exists an ordering of elements of E that satisfies the condition of Algorithm GGA(I) for any such function. The result now follows from Theorem 2.6. □

Lovász [12] considers the following optimization problem over a jump system $\mathbf{J} \subseteq \mathbb{Z}^n$.

Given a box $\mathbf{B} = [a_1,b_1] \otimes \cdots \otimes [a_n,b_n]$, for some $0 \le a_i \le b_i$, $i \in \{1,2,\dots,n\}$, find an $x \in \mathbf{J}$ such that $d(x,\mathbf{B}) = d(\mathbf{J},\mathbf{B}) = \min\{d(y,\mathbf{B}) : y \in \mathbf{J}\}$.

As pointed out in Section 2, $d(y,\mathbf{B})$ is a separable, convex function on $\mathbf{J}$. Lemma 1 in [12] can thus be easily seen to be a special case of Theorem 3.6. By applying Theorem 3.5 to the Δ-matroid corresponding to a given jump system, and using Theorem 2.4, we obtain the following results in [12].

Corollary 6 [12]. *For any jump system* $\mathbf{J} \subseteq \mathbb{Z}^n$ *and any box* $\mathbf{B}, \mathbf{J_B} = \{x : x \in \mathbf{J}; d(x,\mathbf{B}) = d(\mathbf{J},\mathbf{B})\}$ *is a jump system.*

Corollary 7 [12]. *For any jump system* $\mathbf{J} \subseteq \mathbb{Z}^n$ *and any face* $\mathbf{F}$ *of the convex hull of* $\mathbf{J}$, $\mathbf{J} \cap \mathbf{F}$ *is a jump system.*

In fact, if $(X,Y) \in 3^E$ is such that $|X \cap E_i| = a_i$ and $|Y \cap E_i| = b_i$ for all $i = 1,2,\ldots,n$, then it can be readily verified that

$$d(\mathbf{J},\mathbf{B}) = |X| - r(X,Y), \tag{5.4}$$

where $r(\cdot,\cdot)$ is the rank function of the Δ-matroid $(E,\mathfrak{J})$ corresponding to the jump system $\mathbf{J}$. We will now extend slightly the results in [12] on the function $d(\mathbf{J},\mathbf{B})$.

Let $(E,\mathfrak{J})$ be the Δ-matroid corresponding to jump system $\mathbf{J} \subseteq \mathbb{Z}^n$. Let $r(\cdot,\cdot)$ be the rank function of $(E,\mathfrak{J})$. For any $X_1, X_2 \in E$, we say that $X_1 \sim X2$ if and only if $|X_1 \cap E_i| = |X_2 \cap E_i|$ for all $i = 1,2,\ldots,n$. For $(A_1,B_1),(A_2,B_2) \in 3^E, (A_1,B_1) \sim (A_2,B_2)$ if $A_1 \sim A_2$ and $B_1 \sim B_2$. It is easy to see that for $(A_1,B_1),(A_2,B_2) \in 3^E$ such that $(A_1,B_1) \sim (A_2,B_2)$, $r(A_1,B_1) = r(A_2,B_2)$.

Let $g(\cdot,\cdot)$ be an integer-valued, monotonically nondecreasing function on 3^E such that $g(\cdot,\cdot)$ is bisupermodular over $P_{A,B}$ for all $(A,B) \in 3^E$ and for $(A_1,B_1),(A_2,B_2) \in 3^E$ such that $(A_1,B_1) \sim (A_2,B_2)$, $g(A_1,B_1) = g(A_2,B_2)$. (It should be noted that $d(\mathbf{J},\mathbf{B})$ is a function of this type.)

For any $(A,B) \in 3^E$, define

$$S_{A,B} = \{(X,Y) : (X,Y) \sqsubseteq (A,B), g(X,Y) = g(A,B)\}. \tag{5.5}$$

The following lemma is easy to verify. (For detailed proof, the reader is refered to [11].)

Lemma 5.2. $S_{A,B}$ *is a signed-ring family.*

Since $S_{A,B}$ is a signed-ring family, it has a smallest element, which we denote by $\rho(A,B)$. It is easy to see that if $(X,Y) \in S_{A,B}$, then $\rho(X,Y) = \rho(A,B)$. The following lemma can be readily verified.

Lemma 5.3. *For any* $(A,B) \in 3^E$, *let* $(X^0,Y^0) = \rho(A,B)$. *If* $(X,Y) \in S_{A,B}$ *then for any* $i \in \{1,2,\ldots,n\}$,

(i) $|X \cap E_i| < |A \cap E_i| \Rightarrow X^0 \cap E_i = \emptyset$,
(ii) $|Y \cap E_i| < |B \cap E_i| \Rightarrow Y^0 \cap E_i = \emptyset$.

Theorem 5.4. *For any* $(A,B) \in 3^E$, *let* $(X^0,Y^0) = \rho(A,B)$. *Then for any* $i \in \{1,2,\ldots,n\}$,

(i) *either* $(X^0 \cap E_i = \emptyset)$ *or* $(X^0 \cap E_i = A \cap E_i$ *and for any* $X_1 \subset X_2 \subseteq E_i - (A \cup B)$, $g(A \cup X_1,B) < g(A \cup X_2,B))$;
(ii) *either* $(Y^0 \cap E_i = \emptyset)$ *or* $(Y^0 \cap E_i = B \cap E_i$ *and for any* $X_1 \subset X_2 \subseteq E_i - (A \cup B)$, $g(A,B \cup X_1) < g(A,B \cup X_2))$.

Proof. We will prove part (i). Part (ii) follows similarly. It is clear from Lemma 5.3 that either $X^0 \cap E_i = \emptyset$ or $X^0 \cap E_i = A \cap E_i$. Suppose $X^0 \cap E_i = A \cap E_i$ for some $i \in \{1,2,\ldots,n\}$.If for some $X_1 \subset X_2 \subseteq E_i - (A \cup B)$, $g(A \cup X_2,B) = g(A \cup X_1,B)$, then by

Lemma 5.3 and monotonicity of $g(\cdot,\cdot)$,

$$(A - E_i, B) \in S_{A\cup X_2,B} \Longrightarrow (A,B) \in S_{A\cup X_2,B}. \tag{5.6}$$

Hence, $X^0 \cap E_i = \emptyset$, which leads to a contradiction. □

For $(A,B) \in 3^E$, let $\delta(A,B) = |A| - r(A,B)$. Consider $S_{A,B}$ defined as above using the function $g(\cdot,\cdot) = \delta(\cdot,\cdot)$. The following results of Lovász [12] can be obtained from the above theorem.

Corollary 8. *For any $(A,B) \in 3^E$, let $(X^0,Y^0) = \rho(A,B)$. Then for any $i \in \{1,2,\dots,n\}$,*

(i) *either $(X^0 \cap E_i = \emptyset)$ or $(X^0 \cap E_i = A \cap E_i$ and $r(A \cup (E_i - B), B) = r(A,B))$;*
(ii) *either $(Y^0 \cap E_i = \emptyset)$ or $(Y^0 \cap E_i = B \cap E_i$ and for any $X_1 \subset X_2 \subseteq E_i - (A \cup B)$, $r(A, B \cup X_2) < r(A, B \cup X_1))$.*

Proof. Part (ii) follows from Theorem 5.4. We will prove part (i).

From Lemma 5.3, it easily follows that $X^0 \cap E_i = \emptyset$ or $A \cap E_i$, for any $i \in \{1,2,\dots,n\}$. Consider any chain of sets: $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_k = E_i - (A \cup B)$, where, $|X_i| - |X_{i-1}| = 1$ for all $i = 1,2,\dots,k$. If for some $i \geq 1$, $r(A \cup X_i, B) > r(A \cup X_{i-1}, B)$, then $\delta(A \cup X_i, B) = \delta(A \cup X_{i-1}, B)$. From this and monotonicity of $\delta(\cdot,\cdot)$, we get

$$(A - E_i, B) \in S_{A\cup X_2,B} \Longrightarrow (A,B) \in S_{A\cup X_2,B}. \tag{5.7}$$

Therefore, $X^0 \cap E_i = \emptyset$, and this leads to a contradiction. □

Corollary 9. *For any $(A,B) \in 3^E$, let $(X^0,Y^0) = \rho(A,B)$. If for some $i \in \{1,2,\dots,n\}$, $E_i \not\subseteq A \cup B$, then*

(i) *either $X^0 \cap E_i = \emptyset$ or $Y^0 \cap E_i = \emptyset$,*
(ii) *if $Y^0 \cap E_i \neq \emptyset$, then $r(A - E_i, B \cup E_i) = r(A - E_i, B) - |E_i - B|$.*

Proof. Let $X = E_i - (A \cup B)$.

(i) If $X^0 \cap E_i \neq \emptyset$ and $Y^0 \cap E_i \neq \emptyset$, then by Theorem 5.4, $r(A \cup X, B) = r(A,B)$ and $r(A, B \cup X) = r(A,B) - |X|$. Using bisubmodularity of $r(\cdot,\cdot)$, we get

$$2r(A,B) - |X| = r(A \cup X, B) + r(A, B \cup X) \geq 2r(A,B). \tag{5.8}$$

We thus have a contradiction.

(ii) If $Y^0 \cap E_i \neq \emptyset$, then by (i) and Theorem 5.4, $Y^0 \cap E_i = B \cap E_i$ and $X^0 \cap E_i = \emptyset$. Hence, if $(X', Y') = \rho(A - E_i, B)$, then $X' \cap E_i = \emptyset$ and $Y' \cap E_i = B \cap E_i$. Hence, by Theorem 5.4(ii), we get

$$r(A - E_i, B \cup E_i) = r(A - E_i, B) - |E_i - B|. \tag{5.9}$$

This completes the proof. □

The following two results are equivalent to [12, Theorem 8 and Corollary 9].

Theorem 5.5. *Let $(A,B) \in 3^E$ be such that (i) $\delta(A,B) > 0$ and (ii) for all $i \in \{1,2,\dots,n\}$, either $X^0 \cap E_i = \emptyset$ or $Y^0 \cap E_i = \emptyset$, where $(X^0,Y^0) = \rho(A,B)$. Then, there exist $I_1, I_2 \subseteq \{1,2,\dots,n\}$, $I_1 \cap I_2 = \emptyset$, such that*

$$|\cup\{(A\cap E_i) : i \in I_1\}| - |\cup\{(E_i - B) : i \in I_2\}| > r(\cup\{E_i : i \in I_1\}, \cup\{E_i : i \in I_2\}). \tag{5.10}$$

Proof. Let $(X^0,Y^0) = \rho(A,B)$. Let $I_1 = \{i : X^0 \cap E_i \neq \emptyset\}$ and $I_2 = \{i : Y^0 \cap E_i \neq \emptyset\}$. Then,

$$\begin{aligned} 0 &< \delta(X^0,Y^0) \\ &= |\cup\{(A\cap E_i) : i \in I_1\}| - r(\cup\{(A\cap E_i) : i \in I_1\}, \cup\{(B\cap E_i) : i \in I_2\}) \\ &= |\cup\{(A\cap E_i) : i \in I_1\}| - r(\cup\{E_i : i \in I_1\}, \cup\{(B\cap E_i) : i \in I_2\}) \\ &= |\cup\{(A\cap E_i) : i \in I_1\}| - r(\cup\{E_i : i \in I_1\}, \cup\{E_i : i \in I_2\}) \\ &\quad - |\cup\{(E_i - B) : i \in I_2\}|, \end{aligned} \tag{5.11}$$

where the second equality follows from Theorem 5.4(i) and the last one follows from Corollary 9. □

We say that a jump system $\mathbf{J}$ has *constant sum*, if $\sum_i x_i$ is the same for all $x \in \mathbf{J}$.

Corollary 10 [12]. *Let $\mathbf{J} \subseteq \mathbb{Z}_+^n$ be a jump system with constant sum β and let $v \in \mathbb{Z}_+^n$ be such that $\sum_{i=1}^n v_i = \beta$ and $v \notin \mathbf{J}$. Then, there exists an $I \subseteq \{1,2,\dots,n\}$ such that $\sum\{x_i : i \in I\} < \sum\{v_i : i \in I\}$, for all $x \in \mathbf{J}$. That is, every constant sum jump system is a bisubmodular system.*

Proof. If $v_i > \alpha_i$ (where α_i is as defined before) for some $i \in \{1,2,\dots,n\}$, then the result follows. Otherwise, let $(E,\mathfrak{I})$ be the Δ-matroid corresponding to $\mathbf{J}$ and let $X \in \mathfrak{I}_v$. Since $\mathbf{J}$ is a constant sum jump system, $\mathfrak{I}$ is the set of bases of a matroid. Hence, by the result on matroid theory, we have $|X| > r(X,\emptyset)$, which implies that $\delta(X,\emptyset) > 0$. Now, the result follows from Theorem 5.5. □

From Corollaries 7 and 10, Fact 1, and basic convexity theory, we obtain the next result.

Corollary 11. *For any jump system $\mathbf{J} \subseteq \mathbb{Z}^n$, $\{x : x \in \mathbf{J}; \sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i$ for all $y \in \mathbf{J}\}$ is a bisubmodular system.*

Using similar approach, most of the known results on Δ-matroids can be extended to jump systems and simpler proofs can be obtained for most of the known results on jump systems.

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