# THE CONSTRAINED GRAVITY MODEL WITH POWER FUNCTION AS A COST FUNCTION

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A gravity model for trip distribution describes the number of trips between two zones, as a product of three factors, one of the factors is separation or deterrence factor. The deterrence factor is usually a decreasing function of the generalized cost of traveling between the zones, where generalized cost is usually some combination of the travel, the distance traveled, and the actual monetary costs. If the deterrence factor is of the power form and if the total number of origins and destination in each zone is known, then the resulting trip matrix depends solely on parameter, which is generally estimated from data. In this paper, it is shown that as parameter tends to infinity, the trip matrix tends to a limit in which the total cost of trips is the least possible allowed by the given origin and destination totals. If the transportation problem has many cost-minimizing solutions, then it is shown that the limit is one particular solution in which each nonzero flow from an origin to a destination is a product of two strictly positive factors, one associated with the origin and other with the destination. A numerical example is given to illustrate the problem.

## 1. Introduction

The transportation planning process as it is usually carried out consists of a number of stages and at each stage except the first, use is made of the results of previous stages. Trip distribution is one of these stages.

Suppose the number of zones in which trips begin is $N$ and the number of zones in which trips end is $M$. If the number of trips per unit time beginning in origin zone $i$ is $A_i$ $(i = 1, 2, \ldots, N)$ and the number of trips per unit time ending in destination zone $j$ is $B_j$ $(j = 1, 2, \ldots, M)$, then

$$\sum_{i=1}^{N} A_i = \sum_{j=1}^{M} B_j = W, \tag{1.1}$$

where $W$ is the total number of trips being made per unit time. Let us denote $T_{ij}$ the estimated number of trips per unit time, which begins in zone $i$ and ends in zone $j$. The trip distribution process is therefore concerned with obtaining a suitable matrix $(T_{ij})$ of such estimates, which will be called the trip matrix (Wilson [5, 6]).

Any trip matrix $(T_{ij})$ must satisfy the following conditions:

$$
\begin{aligned}
\sum_{j=1}^{M} T_{ij} &= A_i, \quad i = 1, 2, \ldots, N, \\
\sum_{i=1}^{N} T_{ij} &= B_j, \quad j = 1, 2, \ldots, M, \\
T_{ij} \geq 0 \quad &\forall i = 1, 2, \ldots, N; \; j = 1, 2, \ldots, M.
\end{aligned}
\tag{1.2}
$$

Without loss of generality, it is usually assumed that

$$
T_{ij} = R_i S_j f(c_{ij}) \quad \forall i, j,
\tag{1.3}
$$

where $R_i$ is a factor associated with the origin $i = 1, 2, \ldots, N$ and $S_j$ associated with the destination $j = 1, 2, \ldots, M$. $f(*)$ is usually a decreasing function and $c_{ij}$ is the generalized cost of traveling from zone $i$ to zone $j$. By the term generalized cost, we usually mean some combination of distance, time, and direct money charges. If $A_i$, $B_j$, and $c_{ij}$ are known and if the function $f(*)$ is such that $f(c_{ij}) > 0$ for all $i$ and $j$, then it has been shown (Evans [1, 2]) that there is a unique matrix $(T_{ij})$ which satisfies conditions (1.2)-(1.3). Since $A_i$, $B_j$ are strictly positive, so also $R_i$, $S_j$ and hence $T_{ij}$ are strictly positive.

In this paper, we will consider the function $f(c_{ij}) = c_{ij}^{-\alpha} = \exp(-\alpha \log c_{ij})$, where $\alpha$ is a parameter. If this power function is used in the model, then $c_{ij}$ must be strictly positive for all $i$ and $j$.

Therefore, the trip distribution model, which we will consider, is defined by the equations

$$
T_{ij} = R_i S_j \exp\left(-\alpha \log c_{ij}\right) \quad \forall i, j,
\tag{1.4}
$$

$$
\sum_{j=1}^{M} T_{ij} = R_i \sum_{j=1}^{M} S_j \exp\left(-\alpha \log c_{ij}\right) = A_i \quad \forall i,
\tag{1.5}
$$

$$
\sum_{i=1}^{N} T_{ij} = S_j \sum_{i=1}^{N} R_i \exp\left(-\alpha \log c_{ij}\right) = B_j \quad \forall j.
\tag{1.6}
$$

It is called the doubly constrained gravity model with cost function as a power function. If the $A_i$, $B_j$, and $c_{ij}$ are given, as they usually are, then all we need is a value for the parameter $\alpha$ to enable us to solve for the trip matrix $(T_{ij})$. The matrix $(T_{ij})$ can therefore

be regarded as a function of $\alpha$ and the final trip matrix given by model will depend on the value which has been assigned to $\alpha$. From (1.4), we might expect $\alpha$ to measure in some way the extent to which cost is considered when travel decisions are made. Thus we might expect an increase in the value of $\alpha$ to alter the distribution of trips so that the average cost per trip becomes lower. The value of $\alpha$ used in the model is generally estimated from data, for example, from observations of trips being made at present and the corresponding trip costs. This process is called calibration of the model. The parameter is usually chosen so that the mean trip cost in the model is equal to the mean cost of the observed trips.

We suppose that the data, which is available, consists of two matrices, a matrix of the observed number of trips per unit time between each origin and destination, and the matrix of costs applied when these trips were observed. We need to find the $R_i$, the $S_j$, and $\alpha$ such that the resulting matrix $[R_i \ S_j \ \exp(-\alpha \log c_{ij})]$ is in some sense the best possible fit of this model to the data. It is certainly true that for any particular value of $\alpha$, we can find $R_i$ and $S_j$ such that these row and column constraints are satisfied and all the $T_{ij}$ are nonnegative. The problem is therefore to find $\alpha$ such that the solution to the model with these row and column constraints has a mean trip cost equal to the mean trip costing the data. Since the number of trips in the model is equal to the number of trips in the data, this is equivalent to choosing $\alpha$, so that the total cost in the model is the same as the total cost in the data. The structure of this paper is as follows: in Section 2, we will give an alternative minimization formulation of the gravity model to be used in many of the subsequent proofs. It will also help to give a clear understanding of the role of $\alpha$ in the model. Section 3 deals with the total cost of trips as a function of $\alpha$. The results connecting the cost functions will be proved in a more comprehensive form. Section 4 will be concerned with a more detailed discussion of the transportation problem. A numerical example is provided in support of the existence of the problem.

## 2. The gravity model and an equivalent minimization formulation

### 2.1. The doubly constrained gravity model with power function as cost function (see Mazumder and Das [4] and Wilson [7]). The problem is to find a matrix of trips $(T_{ij})$ such that

$$T_{ij} = R_i S_j \exp\left(-\alpha \log c_{ij}\right),$$

$$\sum_{j=1}^{M} T_{ij} = A_i, \quad i = 1, 2, \ldots, N,$$

$$\sum_{i=1}^{N} T_{ij} = B_j, \quad j = 1, 2, \ldots, M, \tag{2.1}$$

where the $A_i$, $B_j$, $c_{ij}$, and the parameter $\alpha$ are assumed known. There is a unique matrix $(T_{ij}^*)$ which satisfies (2.1) and $T_{ij}^*$ is strictly positive for all $i$ and $j$. We will now re-express this as a problem of minimization in which it is required to minimize an objective function $F(*)$ by a choice of $(T_{ij})$ subject to certain constraints. In this context, it is convenient to regard $N \times M$ matrix $(T_{ij})$ as an $NM$-dimensional vector denoted by $\tau$.

**2.2. An equivalent minimization formulation.** We define first the objective function $F(*)$ by

$$F(\tau) = \sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log T_{ij} + \alpha \sum_{i=1}^{N} \sum_{j=1}^{M} (\log c_{ij}) T_{ij}. \tag{2.2}$$

The expression $T_{ij} \log T_{ij}$ is defined for all $T_{ij} > 0$ and we can extend its domain of definition to include the origin by assigning it the value zero at that point. This makes sense since

$$\underset{T_{ij} \to 0+}{Lt} T_{ij} \log T_{ij} = 0. \tag{2.3}$$

Therefore, the function $F(*)$ is denoted for all $\tau$ having $T_{ij} > 0$ for all $i$ and $j$.

Hence the problem is that of minimizing $F(\tau)$ subject to the constraints

$$\sum_{j=1}^{M} T_{ij} = A_i, \quad i = 1, 2, \ldots, N, \tag{2.4}$$

$$\sum_{i=1}^{N} T_{ij} = B_j, \quad j = 1, 2, \ldots, M, \tag{2.5}$$

$$T_{ij} \geq 0 \quad \forall i = 1, 2, \ldots, N; \; j = 1, 2, \ldots, M. \tag{2.6}$$

Obviously, conditions (2.4)–(2.6) define a region in $NM$-dimensional Euclidean space which we will call the feasible region and will be denoted by $\mathscr{D}$. Thus we are required to minimize the objective function $F(\tau)$ over the feasible region $\mathscr{D}$.

To show that this is equivalent to trip distribution problem given in Section 2.1, we form the Lagrangian

$$L(\tau, \alpha, \beta) = \sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log T_{ij} + \alpha \sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} (\log c_{ij}) + \sum_{i=1}^{N} \alpha_i \left( \sum_{j=1}^{M} T_{ij} - A_i \right) + \sum_{j=1}^{M} \beta_i \left( \sum_{i=1}^{N} T_{ij} - B_j \right) \tag{2.7}$$

and take the partial derivative of $L$. Equating this to zero will give the conditions, which $\tau$ must satisfy to be a stationary point of $L$ and hence stationary point of $F(\tau)$ subject to the conditions (2.4)–(2.6). Here one thing to be noted that $\partial L / \partial T_{ij}$ does not exist at the point $T_{ij} = 0$ so that the solution will only be valid if every component of $T_{ij}$ is strictly positive.

Now $\partial L / \partial T_{ij} = 0$ gives $T_{ij} = \exp(-1 - \alpha_i) \exp(-\beta_j) \exp(-\alpha \log c_{ij})$, therefore

$$T_{ij} = R_i S_j \exp\left( -\alpha \log c_{ij} \right) \quad \forall i, j. \tag{2.8}$$

Thus $\tau$ is a stationary point if and only if it satisfies condition (2.8) and constraints (2.4)-(2.5) together with strict inequality of (2.6). Since (2.1) are identical to (2.8), (2.4), and

(2.5), the unique solution $\tau^*$ to (2.1) is the only solution to (2.4)–(2.8) and $T_{ij}^* > 0$ for all $i$ and $j$.

The solution $\tau^*$ to the trip distribution problem of Section 2.1 is therefore the only stationary point of $F(\tau)$ that has strictly positive terms $1/T_{ij}^*$ along the diagonal and zeros everywhere else. It is therefore positive definite and this means that $\tau^*$ is a strict local minimum of $F(\tau)$.

Now we know (Hadley [3]) that if, for a convex function defined on a convex set, there exists a strict local minimum, it is also the unique minimum over the entire convex set. The feasible region $\mathcal{D}$ is certainly convex since it is defined by linear equalities and inequalities. Now we prove a stronger result that $F(*)$ is strictly convex in $\mathcal{D}$.

*Result 2.1.* The function $\sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log T_{ij}$, and hence the function $F(*)$, is strictly convex in $\mathcal{D}$.

There is no difficulty in proving that $\sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log T_{ij}$ is strictly convex on $\mathcal{D}$ since no $\tau$ in the interior of $\mathcal{D}$ can have a zero component, and thus the matrix of second-order derivatives is always positive definite there. This argument cannot however extend to include the boundary of $\mathcal{D}$ since the partial derivatives with respect to $T_{ij}$ do not exist at $T_{ij} = 0$.

Let

$$T(\tau) = \sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log T_{ij},$$

$$C(\tau) = \sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log (c_{ij}). \tag{2.9}$$

Then

$$F(\tau) = T(\tau) + \alpha C(\tau). \tag{2.10}$$

**2.3. About the function $F(*)$.** The function $F(*)$ is the weighted sum of two terms $T(\tau)$ and $C(\tau)$ with weights 1 and $\alpha$, respectively. The negative of the first term can be regarded as a measure of the probability that the trip matrix $(T_{ij})$ will occur in practice if it is assumed that each of the $\gamma$ trips that are made is equally likely to occur between any of the $NM$-possible origin-destination pairs, if $\gamma$ is the total number of trips being made, then the probability that the trip matrix $(T_{ij})$ will occur is

$$\left[ \frac{\gamma!}{\prod_{i,j} (T_{ij})!} \right] \left( \frac{1}{\gamma} \right)^{\gamma}. \tag{2.11}$$

Taking logarithms and using Stirling's approximation for the factorials give

$$-\sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log T_{ij}, \tag{2.12}$$

which is $-T(\tau)$ as required.

Therefore, if we examine maximizing $-T(\tau)$, that is, minimizing $T(\tau)$ subject to origin-destination constraints (2.4)-(2.5), we get a matrix of the form

$$T_{ij} = R_i S_j \quad \forall i, j, \tag{2.13}$$

whence $T_{ij} = A_i B_j / \gamma$ for all $i$ and $j$.

This proportional matrix can be regarded as the most probable matrix under the assumptions that we have made so far.

The second term $C(\tau)$ is the total cost of trips made as a function of trip matrix $(T_{ij})$ and it has weight $\alpha$ which is the function $F(\tau)$ to be minimized. It is obvious that as $\alpha$ increases, the term $\alpha C(\tau)$ dominates the minimization and it is reasonable that as $\alpha$ increases without limit $C(\tau)$, it approaches the minimum possible value $M$ consistent with the constraints (2.4)–(2.6).

### 3. Varying the parameter $\alpha$

**3.1. Some properties of the function $\alpha$.**  Let

$$F(\hat{\tau}(\alpha), \alpha) = \min_{\tau \in D} F(\tau, \alpha) \quad \text{for each } \alpha, \tag{3.1}$$

$$F(\hat{\tau}(\alpha), \alpha) < F(\tau, \alpha) \quad \forall \tau \neq \hat{\tau}(\alpha). \tag{3.2}$$

Since $C(\hat{\tau}(\alpha))$ and $T(\hat{\tau}(\alpha))$ are functions of $\alpha$ alone, we will denote them by $\hat{C}(\tau)$ and $\hat{T}(\tau)$, respectively.

*Result 3.1.*  If $(\log c_{ij})$ is a nontrivial cost matrix, then
   (i) $\hat{C}(\tau)$ is a strictly decreasing function of $\alpha$,
   (ii) $\hat{T}(\tau)$ is a strictly increasing function of $\alpha$ for $\alpha > 0$ and strictly decreasing function of $\alpha$ for $\alpha < 0$.
To prove, suppose $\alpha \neq \beta$ and that neither $\alpha$ nor $\beta$ is zero. Then

$$F(\hat{\tau}(\alpha), \alpha) < F(\hat{\tau}(\beta), \alpha),$$
$$F(\hat{\tau}(\beta), \beta) < F(\hat{\tau}(\alpha), \beta), \tag{3.3}$$

that is,

$$\hat{T}(\alpha) + \alpha \hat{C}(\alpha) < \hat{T}(\beta) + \alpha \hat{C}(\beta),$$
$$\hat{T}(\beta) + \beta \hat{C}(\beta) < \hat{T}(\alpha) + \beta \hat{C}(\alpha). \tag{3.4}$$

Eliminating $\hat{T}$s from (3.4) gives

$$\hat{C}(\beta)(\beta - \alpha) < \hat{C}(\alpha)(\beta - \alpha) \quad \text{from which (i) follows.} \tag{3.5}$$

Similarly, eliminating $\hat{C}$s from (3.4),

$$
\begin{aligned}
\hat{T}(\alpha)(\beta - \alpha) < \hat{T}(\beta)(\beta - \alpha) &: \alpha, \beta > 0, \\
\hat{T}(\alpha)(\beta - \alpha) > \hat{T}(\beta)(\beta - \alpha) &: \alpha, \beta < 0
\end{aligned}
\tag{3.6}
$$

from which (ii) follows.

It is easy to show that $\mathcal{D}$ is closed and bounded, so that $T$ and $C$ are bounded above and below on $\mathcal{D}$ and hence $\hat{C}(\tau)$ and $\hat{T}(\tau)$ are also bounded below and above. Result 3.1 also implies that the limits of $\hat{T}(\tau)$ and $\hat{C}(\tau)$ as $\alpha \to \pm\infty$ exist. Let us now identify the limits.

**3.2. The limits of $\hat{C}(\alpha)$.**  Here we will give a formal definition of the transportation problem and show that the limits of $\hat{C}(\alpha)$ are the minimum and maximum values, which are sought in that problem. Now we consider the minimum transportation problem and the limit of $\hat{C}(\alpha)$ as $\alpha \to +\infty$. The corresponding results for the maximum transportation problem and the limit of $\hat{C}(\alpha)$ as $\alpha \to -\infty$ will follow by symmetry.

In the minimum transportation problem, it is required to minimize the function

$$
C(\tau) = \sum_{i=1}^{N} \sum_{j=1}^{M} T_{ij} \log\left(c_{ij}\right)
\tag{3.7}
$$

by a suitable choice of $\tau$ subject to

$$
\sum_{j=1}^{M} T_{ij} = A_i, \quad i = 1, 2, \ldots, N,
\tag{3.8}
$$

$$
\sum_{i=1}^{N} T_{ij} = B_j, \quad j = 1, 2, \ldots, M,
\tag{3.9}
$$

$$
T_{ij} \geq 0 \quad \forall i = 1, 2, \ldots, N; \ j = 1, 2, \ldots, M.
\tag{3.10}
$$

Conditions (3.8)–(3.10) define the feasible region $\mathcal{D}$ so that it is required to minimize the function $C(\tau)$ over $\mathcal{D}$. Let us denote the minimum possible value for $C(\tau)$ subject to (3.8)–(3.10) is $\lambda$. It is also convex by **Result 2.1** and hence contains either one member or infinitely many. It usually contains just one, but will contain infinitely many if certain relations between the $\log(c_{ij})$ hold.

*Result 3.2.*  The function $\hat{C}(\alpha)$ tends to $\lambda$ as $\alpha$ tends to $+\infty$.

To prove, let $\mathrm{Lim}_{\alpha \to \infty} \hat{C}(\alpha) = \lambda'$, $\lambda' \geq \lambda$, by the definition of $\lambda$ and there exists $\tau_0$ in $\mathcal{D}$ such that $C(\tau_0) = \lambda$.

Suppose $\lambda' > \lambda$. Let $T^*$ and $t^*$ be the upper and lower bounds, respectively, of the function $T$ in $\mathcal{D}$ and choose $\alpha^*$ so that

$$
\alpha^* > \frac{T^* - t^*}{\lambda' - \lambda}.
\tag{3.11}
$$

Then

$$F(\tau_0, \alpha^*) - F(\hat{\tau}(\alpha^*), \alpha^*) = T(\tau_0) - \hat{T}(\alpha^*) + \alpha^* (C(\tau_0) - \hat{C}(\alpha^*))$$

$$\leq T^* - t^* + \alpha^* \{\lambda - \lambda'\} < 0 \quad \text{(using (3.11))}$$

$$(3.12)$$

which contradicts (3.2). Hence $\lambda \not\subset \lambda'$, that is, $\lambda' = \lambda$. This completes the proof of Result 3.2.

## 4. Relevant aspects of the transportation problem and its solution

The constraints (3.8) and (3.9) consist of $(N + M)$ equations of which $(N + M - 1)$ are independent. If $[NM - (N + M - 1)]$ of the variables $T_{ij}$ are set to zero and if the resulting equations in the remaining $(N + M - 1)$ variables can be solved, then we obtain a basic solution. If in addition all the $T_{ij}$ are nonnegative, then condition (3.10) is also satisfied and we have a basic feasible solution. In any feasible, the $[NM - (N + M - 1)]$ variables which were set to zero are known as nonbasic variables and the remaining $(N + M - 1)$ are called basic variables. A basic feasible solution will be called nondegenerate if all its basic variables are strictly positive. It is known that the solution set of the minimum transportation problem contains at least one basic feasible solution. The conditions are expressed in terms of two new sets of variables $u_i$, $i = 1, 2, \ldots, N$, and $v_j$, $j = 1, 2, \ldots, M$, called the dual variables, such that for each basic feasible solution

$$u_i + v_j = \eta_{ij}, \quad \text{where } \eta_{ij} = \log c_{ij}, \ \forall (i, j). \tag{4.1}$$

Since there are $(N + M)$ unknown and only $(N + M - 1)$ equations in (4.1), an arbitrary value is assigned to one of the unknowns, say $u_1 = 0$. It can be shown that a basic feasible solution is a cost-minimizing solution if and only if the corresponding $u_i$ and $v_j$ obtained from (4.1) satisfy the conditions

$$u_i + v_j \leq \eta_{ij} \quad \forall (i, j) \text{ such that } T_{ij} \text{ is nonbasic.} \tag{4.2}$$

It can also be shown that an optimal basic feasible solution is the unique optimal solution if the corresponding $u_i$ and $v_j$ satisfy $u_i + v_j < \eta_{ij}$ for all $(i, j)$ such that $T_{ij}$ is nonbasic. Thus if the optimal solution is not unique, there exist an optimal basic feasible solution and the corresponding $u_i$ and $v_j$ such that

$$u_i + v_j = \eta_{ij} \quad \text{for at least one } (i, j) \text{ for which } T_{ij} \text{ is nonbasic.} \tag{4.3}$$

All the other optimal solutions can be obtained by allowing $T_{ij}$ to be nonzero for all $(i, j)$ such that $u_i + v_j = \eta_{ij}$ and requiring conditions (3.8)–(3.10) to be satisfied as usual. A sufficient condition for nonuniqueness is that there exist a nondegenerate optimal basic feasible solution and the corresponding $u_i$ and $v_j$, where $u_i + v_j = \eta_{ij}$ for at least one $(i, j)$ for which $T_{ij}$ is nonbasic.

We consider the transportation problem when the solution is not unique and suppose that we have an optimal solution, which is also basic. This mean that there exist one or more pairs $(i, j)$ corresponding to nonbasic variables $T_{ij}$ for which $u_i + v_j = \eta_{ij}$.

Let $\Gamma = \{(i,j) : u_i + v_j = \eta_{ij}\}$, then $\Gamma$ contains the pairs just referred to as well as those corresponding to basic variable. It is obvious that $\Gamma$ is independent of the particular basic optimal solution with which we started.

Also we define $\Omega = \{(i,j) : u_i + v_j < \eta_{ij}\}$.

Every optimal solution $\tau$ satisfies

$$T_{ij} = 0 \quad \forall (i,j) \in \Omega \tag{4.4}$$

as well as usual conditions (2.8)–(3.1).

Let the function $\hat{\tau}(\alpha)$ tends to $\tau^*$ as $\alpha$ tends to $\infty$. We can now express $\tau^*$ as the limit of a different function of $\alpha$ as $\alpha$ tends to $\infty$.

We recall that $\hat{\tau}(\alpha)$ is the solution to the gravity model of Section 2.1 with parameter $\alpha$. It satisfies constraints (2.8) and (2.13) which apply to the transportation problem but all its elements $\hat{T}_{ij}(\alpha)$ are strictly positive and of the form $\hat{T}_{ij}(\alpha) = R_i(\alpha)S_j(\alpha)\exp(-\alpha\eta_{ij})$ for all $i$ and $j$.

The limit $\tau^*$ is an optimal solution to the minimum transportation problem and hence

$$T_{ij}^* = 0 \quad \forall (i,j) \text{ in } \Omega. \tag{4.5}$$

Thus for all $(i,j)$ in $\Omega$, $\hat{T}_{ij}(\alpha) \to 0$ as $\alpha \to \infty$.

Neither the solution to the gravity model for any parameter $\alpha$ nor the solution to the transportation problem is affected by the addition of a positive or a negative constant to all the costs in any row or column. Hence without changing the problem, we can subtract $u_i$ from each cost in row i and $v_j$ from each cost in column $j$ of the cost matrix $(\eta_{ij})$ so that

$$\eta_{ij} = 0 \quad \forall (i,j) \text{ in } \Gamma. \tag{4.6}$$

We now define a new function of $\alpha$, $\tilde{\tau}(\alpha)$, such that

$$\tilde{T}_{ij} = \begin{cases} 0 & \forall (i,j) \text{ in } \Omega, \\ R_i(\alpha)S_j(\alpha) & \forall (i,j) \text{ in } \Gamma. \end{cases} \tag{4.7}$$

Clearly $\tilde{\tau}(\alpha)$ tends to $\tau^*$ as $\alpha$ tends to $\infty$.

Introduce a matrix $E = (e_{ij})$, where

$$e_{ij} = \begin{cases} 0 & \forall (i,j) \text{ in } \Omega, \\ 1 & \forall (i,j) \text{ in } \Gamma. \end{cases} \tag{4.8}$$

Then $\tilde{T}_{ij} = R_i(\alpha)S_j(\alpha)e_{ij}$ for all $(i,j)$.

It is easy to see that $\tau^*$ is a solution of the following problem.

*Problem 4.1* ($E,A,B$).  Here $E = (e_{ij})_{N \times M}$ and the problem is defined as

$$T_{ij}^* \geq 0 \quad \forall i,j,$$

$$\sum_{j=1}^{M} T_{ij}^* = A_i \quad \forall i, \tag{4.9}$$

$$\sum_{i=1}^{N} T_{ij}^* = B_j \quad \forall j,$$

where $\widetilde{T}_{ij}^* = \mathrm{Lim}_{\alpha \to \infty} R_i(\alpha) S_j(\alpha) e_{ij}$ for all $i$ and $j$.

The transportation problem is said to be nondegenerate if no partial sum of the $A_i$ is equal to a partial sum of the $B_j$. Therefore, $\tau^*$ must be an interior solution to Problem 4.1 so that $T_{ij}^*$ must be of the form

$$T_{ij}^* = R_i S_j e_{ij} \quad \forall i,j. \tag{4.10}$$

This means that

$$T_{ij}^* = \begin{cases} 0 & \forall (i,j) \text{ in } \Omega, \\ R_i S_j & \forall (i,j) \text{ in } \Gamma. \end{cases} \tag{4.11}$$

But if there is a partial sum of the $A_i$ which is equal to a partial sum of the $B_j$, then the problem is said to be degenerate and in such case, $\tau^*$ is a boundary solution of the problem so that $T_{ij}^*$ has the form

$$T_{ij}^* = R_i S_j \widetilde{e}_{ij} \quad \forall i,j, \tag{4.12}$$

where the matrix $\widetilde{E} = (\widetilde{e}_{ij})_{N \times M}$ is such that

$$\widetilde{e}_{ij} = \begin{cases} 1 & \text{if } T_{ij}^* > 0, \\ 0 & \text{if } T_{ij}^* = 0. \end{cases} \tag{4.13}$$

In either case, we can find $\tau^*$ explicitly by solving Problem 4.1. The matrix $E$ can be found by first using a standard process to find an optimal basic feasible solution to the transportation problem then solving (4.1) for the corresponding $u_i$ and $v_j$, and setting

$$e_{ij} = \begin{cases} 1 & \forall (i,j) \text{ such that } \eta_{ij} = u_i + v_j, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.14}$$

*Numerical example.*  Consider a problem with three origins and three destinations, where $A = (8,7,5)$ and $B = (5,9,6)$, and the cost matrix is

$$\begin{pmatrix} 3 & 3 & 4 \\ 7 & 5 & 4 \\ 5 & 4 & 3 \end{pmatrix}. \tag{4.15}$$

The following matrix is a nondegenerate basic feasible solution to the transportation problem in which the row and column totals are the origin and destination totals, respectively. Solving the equation $\eta_{ij} = u_i + v_j$ for all $(i,j)$ such that $T_{ij}$ is nonzero in this basic feasible solution gives

$$\begin{pmatrix} u_1 = 2 & u_2 = 4 & u_3 = 3 \\ v_1 = 1 & v_2 = 1 & v_3 = 0 \end{pmatrix} \tag{4.16}$$

and these values satisfy the optimality condition $u_i + v_j \leq \eta_{ij}$ for all $(i,j)$ such that $T_{ij}$ is nonzero in the above basic feasible solution. For $i = 3$, and $j = 2$, the optimality condition is satisfied as an equality and because the above optimal basic feasible solution is nondegenerate, this implies that there are many optimal solutions to the transportation problem. Hence the limiting matrix $\tau^*$ must be found by solving **Problem 4.1**, where

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad A = (8,7,5), \qquad B = (5,9,6). \tag{4.17}$$

Since $e_{11} \neq 0$, $\tau^*$ must be an interior solution and is of the form

$$T_{ij}^* = R_i S_j e_{ij} \quad \forall i, j. \tag{4.18}$$

It is easily verified that

$$\tau^* = \begin{pmatrix} 5 & 3 & 0 \\ 0 & 3.5 & 3.5 \\ 0 & 2.5 & 2.5 \end{pmatrix}. \tag{4.19}$$

Since this matrix has the right marginal totals and is of the form

$$T_{ij}^* = R_i S_j e_{ij} \quad \forall i, j, \tag{4.20}$$

where

$$\begin{pmatrix} R_1 = 1 & R_2 = \dfrac{7}{6} & R_3 = \dfrac{5}{6} \\ S_1 = 5 & S_2 = 3 & S_3 = 3 \end{pmatrix}, \tag{4.21}$$

the matrix $\tilde{\tau}(\alpha)$ has been calculated for various values of $\alpha$ using iterative procedure briefly explained in the appendix with the following result:

$$\tilde{\tau}(0) = \begin{pmatrix} 2 & 3.6 & 2.4 \\ 1.75 & 3.15 & 2.1 \\ 1.25 & 2.25 & 1.5 \end{pmatrix},$$

$$\tilde{\tau}(5.0) = \begin{pmatrix} 4.97 & 3.03 & 0.00 \\ 0.00 & 3.49 & 3.51 \\ 0.03 & 2.48 & 2.49 \end{pmatrix}, \tag{4.22}$$

$$\tilde{\tau}(10.0) = \begin{pmatrix} 5 & 3 & 0 \\ 0 & 3.5 & 3.5 \\ 0 & 2.5 & 2.5 \end{pmatrix}.$$

The above result tells us the interesting fact that $\tilde{\tau}(\alpha)$ tends to $\tau^*$ as $\alpha$ increases.

## 5. Conclusion

The solution to the gravity model for trip distribution with given origin and destination totals and cost functions $\mathrm{Exp}(-\alpha \log c_{ij}) = \mathrm{Exp}(-\alpha \eta_{ij})$ varies with the parameter $\alpha$. As $\alpha$ tends to $\infty$, the solution tends to a limit, which is a cost-minimizing solution to the transportation problem of which the marginal totals are the given origin and destination totals. If the transportation problems have many optimal solutions, then the limit is one particular solution so that each nonzero flow is of the form $R_i S_j$ from an origin $i$ to a destination $j$. When the transportation problem is nondegenerate, the only zero flows are possible, which are zeros for optimal solutions. However, if the transportation problem is degenerate, other zero flows may occur.

## Appendix

### Iterative procedure

Consider Problem 4.1.

The procedure starts at stage zero with

$$E_0 = E. \tag{A.1}$$

At each stage $2n$ the row sums of the matrix $E_{2n}$ are made to agree with the $A_i$ and at each stage $(2n+1)$ column sums of the matrix $E_{2n+1}$ are made to agree with the $B_j$. Therefore, row $i$ of $E_{2n}$ must be multiplied by the factor

$$\left( \frac{A_i}{\sum_{j=1}^{M} e_{ij,2n}} \right) \quad \text{to make} \sum_{j=1}^{M} e_{ij,2n} = A_i \ \forall i. \tag{A.2}$$

Similarly, column $j$ of the matrix $E_{2n+1}$ must be multiplied by the factor

$$\left( \frac{B_j}{\sum_{i=1}^{N} e_{ij,2n+1}} \right) \tag{A.3}$$

so that $\sum_{i=1}^{N} e_{ij,2n+1} = B_j$ for all $j$.

The procedure which we apply to solve the above-constrained gravity model is simply an iterative process applied in $[\text{Exp}(-\alpha\eta_{ij}), A, B]$.

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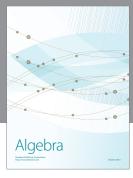*E-mail address*: bablus@rediffmail.com

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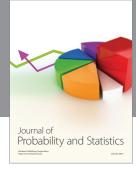*E-mail address*: majumder_sk@yahoo.co.in

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