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Research Article

Callable Russian Options and Their Optimal Boundaries

Atsuo Suzuki[1] and Katsushige Sawaki[2]

[1] *Faculty of Urban Science, Meijo University, 4-3-3 Nijigaoka, Kani, Gifu 509-0261, Japan*
[2] *Nanzan Business School, Nanzan University, 18 Yamazato-cho, Showa-ku, Nagoya 466-8673, Japan*

Correspondence should be addressed to Atsuo Suzuki, atsuo@urban.meijo-u.ac.jp

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We deal with the pricing of callable Russian options. A callable Russian option is a contract in which both of the seller and the buyer have the rights to cancel and to exercise at any time, respectively. The pricing of such an option can be formulated as an optimal stopping problem between the seller and the buyer, and is analyzed as Dynkin game. We derive the value function of callable Russian options and their optimal boundaries.

1. Introduction

For the last two decades there have been numerous papers (see [1]) on valuing American-style options with finite lived maturity. The valuation of such American-style options may often be able to be formulated as optimal stopping or free boundary problems which provide us partial differential equations with specific conditions. One of the difficult problems with pricing such options is finding a closed form solution of the option price. However, there are shortcuts that make it easy to calculate the closed form solution to that option (see [2–4]). Perpetuities can provide us such a shortcut because free boundaries of optimal exercise policies no longer depend on the time.

In this paper, we consider the pricing of Russian options with call provision where the issuer (seller) has the right to call back the option as well as the investor (buyer) has the right to exercise it. The incorporation of call provision provides the issuer with option to retire the obligation whenever the investor exercises his/her option. In their pioneering theoretical studies on Russian options, Shepp and Shiryaev [5, 6] gave an analytical formula for pricing the noncallable Russian option which is one of perpetual American lookback options. The result of this paper is to provide the closed formed solution and optimal boundaries of the callable Russian option with continuous dividend, which is different from the pioneering theoretical paper Kyprianou [2] in the sense that our model has dividend payment.

The paper is organized as follows. In Section 2, we introduce a pricing model of callable Russian options by means of a coupled optimal stopping problem given by Kifer [7]. Section 3 represents the value function of callable Russian options with dividend. Section 4 presents numerical examples to verify analytical results. We end the paper with some concluding remarks and future work.

2. Model

We consider the Black-Scholes economy consisting of two securities, that is, the riskless bond and the stock. Let B_t be the bond price at time t which is given by

$$dB_t = rB_t dt, \quad B_0 > 0, \quad r > 0, \tag{2.1}$$

where r is the riskless interest rate. Let S_t be the stock price at time t which satisfies the stochastic differential equation

$$dS_t = (r-d)S_t dt + \kappa S_t d\widetilde{W}_t, \quad S_0 = x, \tag{2.2}$$

where d and $\kappa > 0$ are constants, d is dividend rate, and $\widetilde{W}_t$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \widetilde{P})$. Solving (2.2) with the initial condition $S_0 = x$ gives

$$S_t = x\exp\left\{\left(r - d - \frac{1}{2}\kappa^2\right)t + \kappa\widetilde{W}_t\right\}. \tag{2.3}$$

Define another probability measure $\widehat{P}$ by

$$\frac{d\widehat{P}}{d\widetilde{P}} = \exp\left(\kappa\widetilde{W}_t - \frac{1}{2}\kappa^2 t\right). \tag{2.4}$$

Let

$$\widehat{W}_t = \widetilde{W}_t - \kappa t, \tag{2.5}$$

where $\widehat{W}_t$ is a standard Brownian motion with respect to $\widehat{P}$. Substituting (2.5) into (2.2), we get

$$dS_t = \left(r - d + \kappa^2\right)S_t dt + \kappa S_t d\widehat{W}_t. \tag{2.6}$$

Solving the above equation, we obtain

$$S_t(x) = x\exp\left\{\left(r - d + \frac{1}{2}\kappa^2\right)t + \kappa\widehat{W}_t\right\}. \tag{2.7}$$

Russian option was introduced by Shepp and Shiryaev [5, 6] and is the contract that only the buyer has the right to exercise it. On the other hand, a callable Russian option is the contract that the seller and the buyer have both the rights to cancel and to exercise it at any time, respectively. Let σ be a cancel time for the seller and τ be an exercise time for the buyer. We set

$$\Psi_t(\psi) \equiv \frac{\max\left(\psi x, \sup_{0\le u\le t} S_u\right)}{S_t}, \quad \psi \ge 1. \tag{2.8}$$

When the buyer exercises the contract, the seller pay $\Psi_\tau(\psi)$ to the buyer. When the seller cancels it, the buyer receives $\Psi_\sigma(\psi) + \delta$. We assume that seller's right precedes buyer's one when $\sigma = \tau$. The payoff function of the callable Russian option is given by

$$R(\sigma,\tau) = \left(\Psi_\sigma(\psi) + \delta\right)1_{\{\sigma<\tau\}} + \Psi_\tau(\psi)1_{\{\tau\le\sigma\}}, \tag{2.9}$$

where δ is the penalty cost for the cancel and a positive constant.

Let $\mathcal{T}_{0,\infty}$ be the set of stopping times with respect to filtration $\mathcal{F}$ defined on the nonnegative interval. Letting α and ψ be some given parameters satisfying $\alpha > 0$ and $\psi \ge 1$, the value function of the callable Russian option $V(\psi)$ is defined by

$$V(\psi) = \inf_{\sigma\in\mathcal{T}_{0,\infty}} \sup_{\tau\in\mathcal{T}_{0,\infty}} \widehat{E}\left[e^{-\alpha(\sigma\wedge\tau)} R(\sigma,\tau)\right], \quad \alpha > 0. \tag{2.10}$$

The infimum and supremum are taken over all stopping times σ and τ, respectively.

We define two sets A and B as

$$\begin{aligned} A &= \left\{\psi \in \mathbf{R}^+ \mid V(\psi) = \psi + \delta\right\}, \\ B &= \left\{\psi \in \mathbf{R}^+ \mid V(\psi) = \psi\right\}. \end{aligned} \tag{2.11}$$

A and B are called the seller's cancel region and the buyer's exercise region, respectively. Let σ_A^ψ and τ_B^ψ be the first hitting times that the process $\Psi_t(\psi)$ is in the region A and B, that is,

$$\begin{aligned} \sigma_A^\psi &= \inf\left\{t > 0 \mid \Psi_t(\psi) \in A\right\}, \\ \tau_B^\psi &= \inf\left\{t > 0 \mid \Psi_t(\psi) \in B\right\}. \end{aligned} \tag{2.12}$$

Lemma 2.1. *Assume that $d - (1/2)\kappa^2 - 2r < 0$. Then, one has*

$$\lim_{t\to\infty} e^{-rt}\Psi_t(\psi) = 0. \tag{2.13}$$

Proof. First, suppose that $\max(\psi x, \sup S_u) = \psi x$. Then, it holds

$$\lim_{t\to\infty} e^{-rt} S_t^{-1} = \lim_{t\to\infty} \exp\left\{ -\kappa\widetilde{W}_t + \left(d + \frac{1}{2}\kappa^2 - 2r\right)t\right\} = 0. \tag{2.14}$$

Next, suppose that $\max(\psi x, \sup S_u) = \sup S_u$. By the same argument as Karatzas and Shreve [1, page 65], we obtain

$$\lim_{t\to\infty} \sup S_u = x \exp\left\{\kappa \cdot \sup_{0<u<\infty}\left(\widetilde{W}_u + \frac{r-d}{\kappa} - \frac{1}{2}\kappa^2\right)\right\} = x\exp\{\kappa W^*\}, \tag{2.15}$$

where W^* is the standard Brownian motion which attains the supremum in (2.15). Therefore, it follows that

$$\lim_{t\to\infty} e^{-rt}\frac{\sup S_u}{S_t} = 0. \tag{2.16}$$

The proof is complete. □

By this lemma, we may apply Proposition 3.3 in Kifer [7]. Therefore, we can see that the stopping times $\widehat{\sigma}^{\psi} = \sigma_A^{\psi}$ and $\widehat{\tau}^{\psi} = \tau_B^{\psi}$ attain the infimum and the supremum in (2.10). Then, we have

$$V(\psi) = \widehat{E}\left[e^{-\alpha(\widehat{\sigma}^{\psi}\wedge\widehat{\tau}^{\psi})}R(\widehat{\sigma}^{\psi}, \widehat{\tau}^{\psi})\right]. \tag{2.17}$$

And $V(\psi)$ satisfies the inequalities

$$\psi \le V(\psi) \le \psi + \delta, \tag{2.18}$$

which provides the lower and the upper bounds for the value function of the callable Russian option. Let $V_R(\psi)$ be the value function of Russian option. And we know $V(\psi) \le V_R(\psi)$ because the seller as a minimizer has the right to cancel the option. Moreover, it is clear that $V(\psi)$ is increasing in ψ and x.

Should the penalty cost δ be large enough, it is optimal for the seller not to cancel the option. This raises a question how large such a penalty cost should be. The following lemma is to answer the question.

Lemma 2.2. *Set $\delta^* = V(1) - 1$. If $\delta \ge \delta^*$, the seller never cancels. Therefore, callable Russian options are reduced to Russian options.*

Proof. We set $h(\psi) = V(\psi) - \psi - \delta$. $h'(\psi) = V'(\psi) - 1 < 0$. Because we know $h(1) = V(1) - 1 - \delta = \delta^* - \delta < 0$ by the condition $\delta \ge \delta^*$, we have $h(\psi) < 0$, that is, $V(\psi) < \psi + \delta$ holds. By using the relation $V(\psi) \le V_R(\psi)$, we obtain $V(\psi) < \psi + \delta$, that is, it is optimal for the seller not to cancel. Therefore, the seller never cancels the contract for $\delta \ge \delta^*$. □

Lemma 2.3. *Suppose $r > d$. Then, the function $V(\psi)$ is Lipschitz continuous in ψ. And it holds*

$$0 \le \frac{dV(\psi)}{d\psi} \le 1. \tag{2.19}$$

Proof. Set

$$J^{\psi}(\hat{\sigma}^{\psi},\hat{\tau}^{\phi}) = \hat{E}\left[e^{-\alpha(\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\psi})}R(\hat{\sigma}^{\psi},\hat{\tau}^{\psi})\right]. \tag{2.20}$$

Replacing the optimal stopping times $\hat{\sigma}^{\phi}$ and $\hat{\tau}^{\psi}$ from the nonoptimal stopping times $\hat{\sigma}^{\psi}$ and $\hat{\tau}^{\phi}$, we have

$$\begin{aligned} V(\psi) &\geq J^{\psi}(\hat{\sigma}^{\psi},\hat{\tau}^{\phi}),\\ V(\phi) &\leq J^{\phi}(\hat{\sigma}^{\psi},\hat{\tau}^{\phi}), \end{aligned} \tag{2.21}$$

respectively. Note that $z_1^+ - z_2^+ \leq (z_1 - z_2)^+$. For any $\phi > \psi$, we have

$$\begin{aligned} 0 &\leq V(\phi) - V(\psi)\\ &\leq J^{\phi}(\hat{\sigma}^{\psi},\hat{\tau}^{\phi}) - J^{\psi}(\hat{\sigma}^{\psi},\hat{\tau}^{\phi})\\ &= \hat{E}\left[e^{-\alpha(\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\phi})}\left(\Psi_{\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\phi}}(\phi) - \Psi_{\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\phi}}(\psi)\right)\right]\\ &= \hat{E}\left[e^{-\alpha(\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\phi})}H^{-1}_{\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\phi}}\left((\phi - \sup H_u)^+ - (\psi - \sup H_u)^+\right)\right]\\ &\leq (\phi-\psi)\hat{E}\left[e^{-\alpha(\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\phi})}H^{-1}_{\hat{\sigma}^{\psi}\wedge\hat{\tau}^{\phi}}\right], \end{aligned} \tag{2.22}$$

where $H_t = \exp\{(r - d + (1/2)\kappa^2)t + \kappa\widehat{W}_t\}$. Since the above expectation is less than 1, we have

$$0 \leq V(\phi) - V(\psi) \leq \phi - \psi. \tag{2.23}$$

This means that V is Lipschitz continuous in ψ, and (2.19) holds. □

By regarding callable Russian options as a perpetual double barrier option, the optimal stopping problem can be transformed into a constant boundary problem with lower and upper boundaries. Let $\tilde{B} = \{\psi \in \mathbf{R}^+ \mid V_R(\psi) = \psi\}$ be the exercise region of Russian option. By the inequality $V(\psi) \leq V_R(\psi)$, it holds $B \supset \tilde{B} \neq \emptyset$. Consequently, we can see that the exercise region B is the interval $[l_*, \infty)$. On the other hand, the seller minimizes $R(\sigma,\tau)$ and it holds $\Psi_t(\psi) \geq \Psi_0(\psi) = \psi \geq 1$. From this, it follows that the seller's optimal boundary A is a point $\{1\}$. The function $V(\psi)$ is represented by

$$V(\psi) = \begin{cases} V_{\psi}(l_*), & 1 \leq \psi \leq l_*,\\ \psi, & \psi \geq l_*, \end{cases} \tag{2.24}$$

where

$$V_{\psi}(l_*) = (1+\delta)\hat{E}\left[e^{-\alpha\sigma_1^{\psi}}1_{\{\sigma_1^{\psi} < \tau^{\psi}_{[l_*,\infty)}\}}\right] + l\hat{E}\left[e^{-\alpha\tau^{\psi}_{[l_*,\infty)}}1_{\{\tau^{\psi}_{[l_*,\infty)} \leq \sigma_1^{\psi}\}}\right]. \tag{2.25}$$

In order to calculate (2.25), we prepare the following lemma.

Lemma 2.4. *Let σ_a^x and τ_b^x be the first hitting times of the process $S_t(x)$ to the points $\{a\}$ and $\{b\}$. Set $\nu = (r-d)/\kappa - (1/2)\kappa$, $\eta_1 = (1/\kappa)(\sqrt{\nu^2+2\alpha}+\nu)$, and $\eta_2 = (1/\kappa)(\sqrt{\nu^2+2\alpha}-\nu)$. Then for $a < x < b$, one has*

$$\widetilde{E}\left[e^{-\alpha\sigma_a^x}1_{\{\sigma_a^x<\tau_b^x\}}\right] = \frac{(b/x)^{\eta_1}-(x/b)^{\eta_2}}{(b/a)^{\eta_1}-(a/b)^{\eta_2}}, \tag{2.26}$$

$$\widetilde{E}\left[e^{-\alpha\tau_b^x}1_{\{\tau_b^x<\sigma_a^x\}}\right] = \frac{(a/x)^{\eta_1}-(x/a)^{\eta_2}}{(a/b)^{\eta_1}-(b/a)^{\eta_2}}. \tag{2.27}$$

Proof. First, we prove (2.26). Define

$$L_t = \exp\left(-\frac{1}{2}\nu^2 t - \nu\widetilde{W}_t\right). \tag{2.28}$$

We define $\widehat{P}$ as $d\widehat{P} = L_T d\widetilde{P}$. By Girsanov's theorem, $\widehat{W}_t \equiv \widetilde{W}_t + \nu t$ is a standard Brownian motion under the probability measure $\widehat{P}$. Let T_{ρ_1} and T_{ρ_2} be the first time that the process $\widehat{W}_t$ hits ρ_1 or ρ_2, respectively, that is,

$$\begin{aligned} T_{\rho_1} &= \inf\{t>0 \mid \widehat{W}_t = \rho_1\}, \\ T_{\rho_2} &= \inf\{t>0 \mid \widehat{W}_t = \rho_2\}. \end{aligned} \tag{2.29}$$

Since we obtain $\log S_t(x) = \log x + \kappa\widehat{W}_t$ from $S_t(x) = x\exp(\kappa\widehat{W}_t)$, we have

$$\begin{aligned} \sigma_a^x &= T_{\rho_1}, \quad \text{a.s.}, \ \rho_1 = \frac{1}{\kappa}\log\frac{a}{x}, \\ \tau_b^x &= T_{\rho_2}, \quad \text{a.s.}, \ \rho_2 = \frac{1}{\kappa}\log\frac{b}{x}, \\ L_{T_{\rho_1}}^{-1} &= \exp\left(\frac{1}{2}\nu^2 T_{\rho_1} + \nu\widetilde{W}_{T_{\rho_1}}\right) \\ &= \exp\left(-\frac{1}{2}\nu^2 T_{\rho_1} + \nu\widehat{W}_{T_{\rho_1}}\right) \\ &= \exp\left(-\frac{1}{2}\nu^2 T_{\rho_1} + \nu\rho_1\right). \end{aligned} \tag{2.30}$$

Therefore, we have

$$\begin{aligned} \widetilde{E}\left[e^{-\alpha\sigma_a^x}1_{\{\sigma_a^x<\tau_b^x\}}\right] &= \widetilde{E}\left[e^{-\alpha T_{\rho_1}}1_{\{T_{\rho_1}<T_{\rho_2}\}}\right] \\ &= \widehat{E}\left[\exp\left(-\frac{1}{2}\nu^2 T_{\rho_1} + \nu\rho_1\right)e^{-\alpha T_{\rho_1}}1_{\{T_{\rho_1}<T_{\rho_2}\}}\right] \\ &= e^{\nu\rho_1}\widehat{E}\left[\exp\left\{-\left(\alpha+\frac{1}{2}\nu^2\right)T_{\rho_1}\right\}1_{\{T_{\rho_1}<T_{\rho_2}\}}\right]. \end{aligned} \tag{2.31}$$

From Karatzas and Shreve [8, Exercise 8.11, page 100], we can see that

$$\widehat{E}\left[\exp\left\{-\left(\alpha+\frac{1}{2}\nu^2\right)T_{\rho_1}\right\}1_{\{T_{\rho_1}<T_{\rho_2}\}}\right]=\frac{\sinh\rho_2\sqrt{\nu^2+2\alpha}}{\sinh(\rho_2-\rho_1)\sqrt{\nu^2+2\alpha}}. \tag{2.32}$$

Therefore, we obtain

$$\begin{aligned}&\widetilde{E}\left[e^{-\alpha\sigma_a^x}1_{\{\sigma_a^x<\tau_b^x\}}\right]\\&\quad=\frac{\sinh\rho_2\sqrt{\nu^2+2\alpha}}{\sinh(\rho_2-\rho_1)\sqrt{\nu^2+2\alpha}}\frac{e^{\nu\rho_2}}{e^{\nu(\rho_2-\rho_1)}}=\frac{e^{\kappa\rho_2\gamma_1}-e^{-\kappa\rho_2\gamma_2}}{e^{\kappa(\rho_2-\rho_1)\gamma_1}-e^{-\kappa(\rho_2-\rho_1)\gamma_2}}=\frac{(b/x)^{\gamma_1}-(x/b)^{\gamma_2}}{(b/a)^{\gamma_1}-(a/b)^{\gamma_2}}.\end{aligned} \tag{2.33}$$

We omit the proof of (2.27) since it is similar to that of (2.26). □

We study the boundary point l_* of the exercise region for the buyer. For $1<\psi<l<\infty$, we consider the function $V(\psi,l)$. It is represented by

$$V(\psi,l)=\begin{cases}V_\psi(l), & 1\le\psi\le l,\\ \psi, & \psi\ge l.\end{cases} \tag{2.34}$$

The family of the functions $\{V(\psi,l),1<\psi<l\}$ satisfies

$$V(\psi)=V(\psi,l_*)=\sup_{1<\psi<l}V(\psi,l). \tag{2.35}$$

To get an optimal boundary point l_*, we compute the partial derivative of $V(\psi,l)$ with respect to l, which is given by the following lemma.

Lemma 2.5. *For any* $1<\psi<l$, *one has*

$$\frac{\partial V}{\partial l}(\psi,l)=\frac{\psi^{\eta_2}-\psi^{-\eta_1}}{l(l^{\eta_1}-l^{-\eta_2})^2}l^{\eta_1}l^{-\eta_2}\left\{(1-\eta_2)l^{\eta_1+1}-(1+\eta_1)l^{-\eta_2+1}+(1+\delta)(\eta_1+\eta_2)\right\}. \tag{2.36}$$

Proof. First, the derivative of the first term is

$$\begin{aligned}\frac{\partial}{\partial l}\left(\frac{(l/\psi)^{\eta_1}-(\psi/l)^{\eta_2}}{l^{\eta_1}-l^{-\eta_2}}\right)&=\frac{1}{(l^{\eta_1}-l^{-\eta_2})^2}\left\{\left(\eta_1\left(\frac{l}{\psi}\right)^{\eta_1-1}\frac{1}{\psi}+\eta_2\left(\frac{\psi}{l}\right)^{\eta_2}\frac{1}{l}\right)(l^{\eta_1}-l^{-\eta_2})\right.\\&\qquad\left.-\left(\left(\frac{l}{\psi}\right)^{\eta_1}-\left(\frac{\psi}{l}\right)^{\eta_2}\right)(\eta_1l^{\eta_1-1}+\eta_2l^{-\eta_2-1})\right\}\\&=\frac{1}{l(l^{\eta_1}-l^{-\eta_2})^2}(\eta_1+\eta_2)\left\{l^{\eta_1}\left(\frac{\psi}{l}\right)^{\eta_2}-l^{-\eta_2}\left(\frac{l}{\psi}\right)^{\eta_1}\right\}\\&=\frac{1}{l(l^{\eta_1}-l^{-\eta_2})^2}(\eta_1+\eta_2)l^{\eta_1}l^{-\eta_2}(\psi^{\eta_2}-\psi^{-\eta_1}).\end{aligned} \tag{2.37}$$

Next, the derivative of the second term is

$$\frac{\partial}{\partial l}\left(\frac{l}{l^{\eta_2}-l^{-\eta_1}}\right)=\frac{(1-\eta_2)l^{\eta_2}-(1+\eta_1)l^{-\eta_1}}{(l^{\eta_2}-l^{-\eta_1})^2}=\frac{(1-\eta_2)l^{\eta_1}-(1+\eta_1)l^{-\eta_2}}{(l^{\eta_1}-l^{-\eta_2})^2}l^{\eta_1-\eta_2},\tag{2.38}$$

where the last equality follows from the relation

$$(l^{\eta_2}-l^{-\eta_1})l^{\eta_1-1}l^{-\eta_2+1}=l^{\eta_1}-l^{-\eta_2}.\tag{2.39}$$

After multiplying (2.37) by $(1+\delta)$ and (2.38) by $\psi^{\eta_2}-\psi^{-\eta_1}$, we obtain (2.36). □

We set

$$f(l)=(1-\eta_2)l^{\eta_1+1}-(1+\eta_1)l^{-\eta_2+1}+(1+\delta)(\eta_1+\eta_2).\tag{2.40}$$

Since $f(1)=\delta(\eta_1+\eta_2)>0$ and $f(\infty)=-\infty$, the equation $f(l)=0$ has at least one solution in the interval $(1,\infty)$. We label all real solutions as $1<l_n<l_{n-1}<\cdots<l_1<\infty$. Then, we have

$$\frac{\partial V}{\partial l}(\psi,l)\Big|_{l=l_i}=0,\quad i=1,\ldots,n\ \forall\psi.\tag{2.41}$$

Then $l_*=l_1$ attains the supremum of $V(\psi,l)$. In the following, we will show that the function $V(\psi)$ is convex and satisfies smooth-pasting condition.

Lemma 2.6. *$V(\psi)$ is a convex function in ψ.*

Proof. From (2.50), V satisfies

$$\frac{1}{2}\kappa^2\psi^2\frac{d^2V}{d\psi^2}=-(r-d)\psi\frac{dV}{d\psi}+\alpha V(\psi).\tag{2.42}$$

If $r\le d$, we get $d^2V/d\psi^2>0$. Next assume that $r>d$. We consider function $\widetilde{V}(\psi)=V(-\psi)$ for $\psi<0$. Then,

$$\frac{1}{2}\kappa^2\psi^2\frac{d^2\widetilde{V}}{d\psi^2}-(r-d)\psi\frac{d\widetilde{V}}{d\psi}-r\widetilde{V}=\frac{1}{2}\kappa^2\psi^2\frac{d^2V}{d\psi^2}+(r-d)\psi\frac{dV}{d\psi}-rV=0.\tag{2.43}$$

Since we find that $d^2\widetilde{V}/d\psi^2>0$ from the above equation, $\widetilde{V}$ is a convex function. It follows from this the fact that V is a convex function. □

Lemma 2.7. *$V(\psi)$ satisfies*

$$\frac{dV}{d\psi}(l_*^-) = \frac{dV}{d\psi}(l_*^+) = 1. \tag{2.44}$$

Proof. Since $V(\psi) = \psi$ for $\psi > l_*$, it holds $(dV/d\psi)(l_*^+) = 1$. For $1 \le \psi < l_*$, we derivative (2.47):

$$\begin{aligned}
\frac{dV}{d\psi} &= \frac{l}{l^{\eta_2} - l^{-\eta_1}}\left(\eta_2 \psi^{\eta_2 - 1} + \eta_1 l^{-\eta_1 - 1}\right) + \frac{1+\delta}{l^{\eta_1} - l^{-\eta_2}}\left(-\eta_1\left(\frac{l}{\psi}\right)^{\eta_1}\frac{1}{\psi} - \eta_2\left(\frac{\psi}{l}\right)^{\eta_2}\frac{1}{\psi}\right) \\
&= \frac{1}{\psi(l^{\eta_1} - l^{-\eta_2})}\left\{ l^{\eta_1 - \eta_2 + 1}\left(\eta_2\psi^{\eta_2} + \eta_1\psi^{-\eta_1}\right) - (1+\delta)\left(\eta_1\left(\frac{l}{\psi}\right)^{\eta_1} + \eta_2\left(\frac{\psi}{l}\right)^{\eta_2}\right)\right\} \\
&= \frac{1}{\psi(l^{\eta_1} - l^{-\eta_2})}\left\{ \eta_2\left(\frac{\psi}{l}\right)^{\eta_2} l^{\eta_1 + 1} + \eta_1\left(\frac{l}{\psi}\right)^{\eta_1} l^{-\eta_2 + 1} - (1+\delta)\left(\eta_1\left(\frac{l}{\psi}\right)^{\eta_1} + \eta_2\left(\frac{\psi}{l}\right)^{\eta_2}\right)\right\}.
\end{aligned} \tag{2.45}$$

Therefore, we get

$$\begin{aligned}
\frac{dV}{d\psi}(l_*) - 1 &= \frac{1}{\left(l_*^{\eta_1+1} - l_*^{1-\eta_2}\right)}\left\{\eta_2 l_*^{\eta_1+1} + \eta_1 l_*^{-\eta_2+1} - (1+\delta)(\eta_1+\eta_2) - \left(l_*^{\eta_1+1} - l_*^{-\eta_2+1}\right)\right\} \\
&= \frac{1}{\left(l_*^{\eta_1} - l_*^{-\eta_2}\right)}\left\{(\eta_2 - 1) l_*^{\eta_1+1} + (\eta_1+1) l_*^{-\eta_2+1} - (1+\delta)(\eta_1+\eta_2)\right\} \\
&= \frac{1}{\left(l_*^{\eta_1} - l_*^{-\eta_2}\right)} f(l_*) \\
&= 0.
\end{aligned} \tag{2.46}$$

This completes the proof. □

Therefore, we obtain the following theorem.

Theorem 2.8. *The value function of callable Russian option $V(\psi)$ is given by*

$$V(\psi) = \begin{cases} (1+\delta)\dfrac{(l_*/\psi)^{\eta_1} - (\psi/l_*)^{\eta_2}}{l_*^{\eta_1} - l_*^{-\eta_2}} + l_*\dfrac{\psi^{\eta_2} - \psi^{-\eta_1}}{l_*^{\eta_2} - l_*^{-\eta_1}}, & 1 \le \psi \le l_*, \\ \psi, & \psi \ge l_*. \end{cases} \tag{2.47}$$

And the optimal stopping times are

$$\begin{aligned}
\hat{\sigma}^{\psi} &= \inf\{t > 0 \mid \Psi_t(\psi) = 1\}, \\
\hat{\tau}^{\psi} &= \inf\{t > 0 \mid \Psi_t(\psi) \ge l_*\}.
\end{aligned} \tag{2.48}$$

The optimal boundary for the buyer l_* is the solution in $(1,\infty)$ to $f(l)=0$, where

$$f(l) = (1-\eta_2)l^{\eta_1+1} - (1+\eta_1)l^{-\eta_2+1} + (1+\delta)(\eta_1+\eta_2). \tag{2.49}$$

We can get (2.47) by another method. For $1 < \psi < l$, the function $V(\psi)$ satisfies the differential equation

$$\frac{1}{2}\kappa^2\psi^2\frac{d^2V}{d\psi^2} + (r-d)\psi\frac{dV}{d\psi} - \alpha V(\psi) = 0. \tag{2.50}$$

Also, we have the boundary conditions as follows:

$$V(1) = C_1 + C_2 = 1+\delta, \tag{2.51}$$

$$V(l) = C_1 l^{\lambda_1} + C_2 l^{\lambda_2} = l, \tag{2.52}$$

$$V'(l) = C_1\lambda_1 l^{\lambda_1-1} + C_2\lambda_2 l^{\lambda_2-1} = 1. \tag{2.53}$$

The general solution to (2.50) is represented by

$$V(\psi) = C_1\psi^{\lambda_1} + C_2\psi^{\lambda_2}, \tag{2.54}$$

where C_1 and C_2 are constants. Here, λ_1 and λ_2 are the roots of

$$\frac{1}{2}\kappa^2\lambda^2 + \left(r-d-\frac{1}{2}\kappa^2\right)\lambda - \alpha = 0. \tag{2.55}$$

Therefore, λ_1, λ_2 are

$$\lambda_{1,2} = \frac{\pm\sqrt{\nu^2+2\alpha}-\nu}{\kappa}. \tag{2.56}$$

From conditions (2.51) and (2.52), we get

$$C_1 = \frac{l-(\delta+1)l^{\lambda_2}}{l^{\lambda_1}-l^{\lambda_2}}, \qquad C_2 = \frac{(\delta+1)l^{\lambda_1}-l}{l^{\lambda_1}-l^{\lambda_2}}. \tag{2.57}$$

And from (2.57) and (2.53), we have

$$(1-\eta_2)l^{\eta_1+1} - (1+\eta_1)l^{-\eta_2+1} + (1+\delta)(\eta_1+\eta_2) = 0. \tag{2.58}$$

Substituting (2.57) into (2.54), we can obtain (2.47).

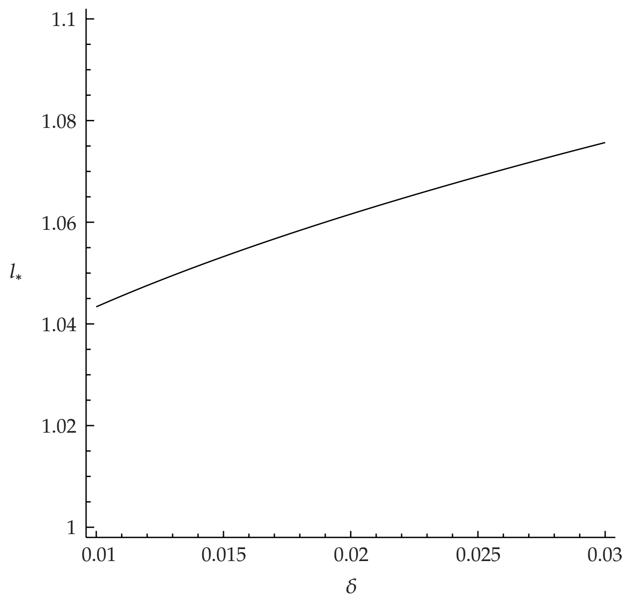

Figure 1: Optimal boundary for the buyer.

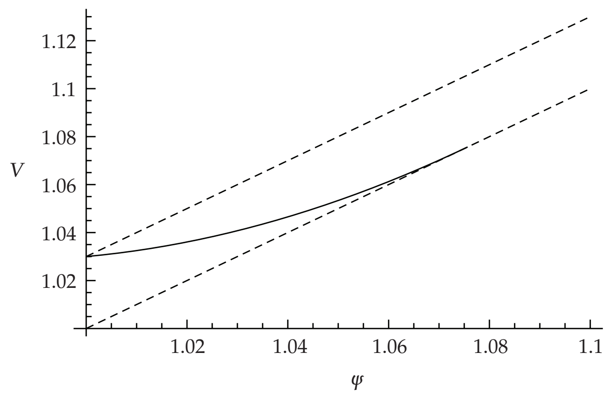

Figure 2: The value function $V(\psi)$ ($\delta = 0.03$).

3. Numerical Examples

In this section, we present some numerical examples which show that theoretical results are varied and some effects of the parameters on the price of the callable Russian option. We use the values of the parameters as follows: $\alpha = 0.5$, $r = 0.1$, $d = 0.09$, $\kappa = 0.3$, $\delta = 0.03$.

Figure 1 shows an optimal boundary for the buyer as a function of penalty costs δ, which is increasing in δ. Figures 2 and 3 show that the price of the callable Russian option has the low and upper bounds and is increasing and convex in ψ. Furthermore, we know that $V(\psi)$ is increasing in δ. Figure 4 demonstrates that the price of the callable Russian option with dividend is equal to or less than the one without dividend. Table 1 presents the values of the optimal boundaries for several combinations of the parameters.

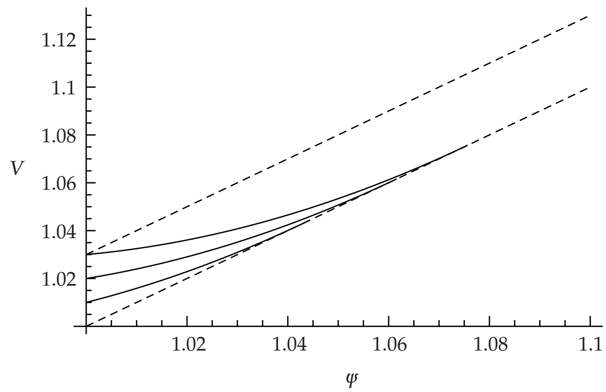

Figure 3: The value function $V(\psi)$ $(\delta = 0.01, 0.02, 0.03)$.

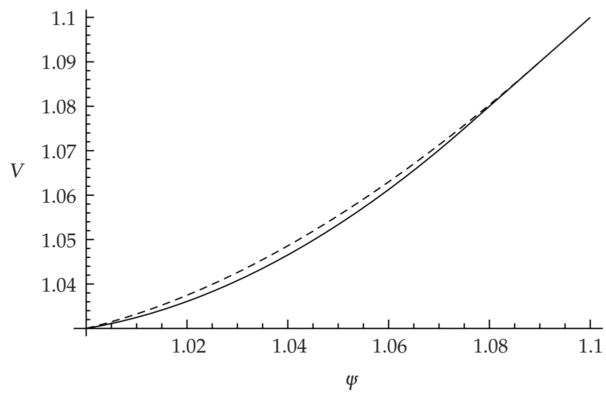

Figure 4: Real line with dividend; dash line without dividend.

4. Concluding Remarks

In this paper, we considered the pricing model of callable Russian options, where the stock pays continuously dividend. We derived the closed-form solution of such a Russian option as well as the optimal boundaries for the seller and the buyer, respectively. It is of interest to note that the price of the callable Russian option with dividend is not equal to the one as dividend value d goes to zero. This implicitly insist that the price of the callable Russian option without dividend is not merely the limit value of the one as if dividend vanishes as d goes to zero. We leave the rigorous proof for this question to future research. Further research is left for future work. For example, can the price of callable Russian options be decomposed into the sum of the prices of the noncallable Russian option and the callable discount? If the callable Russian option is finite lived, it is an interesting problem to evaluate the price of callable Russian option as the difference between the existing price formula and the premium value of the call provision.

Table 1: Penalty δ, interest rate r, dividend rate d, volatility κ, discount factor α, and the optimal boundary for the buyer l_*.

δ	r	d	κ	α	l_*
0.01	0.1	0.09	0.3	0.5	1.04337
0.02	0.1	0.09	0.3	0.5	1.0616
0.03	0.1	0.09	0.3	0.5	1.07568
0.03	0.2	0.09	0.3	0.5	1.08246
0.03	0.3	0.09	0.3	0.5	1.09228
0.03	0.4	0.09	0.3	0.5	1.10842
0.03	0.5	0.09	0.3	0.5	1.14367
0.03	0.1	0.01	0.3	0.5	1.08092
0.03	0.1	0.05	0.3	0.5	1.07813
0.03	0.1	0.1	0.3	0.5	1.07511
0.03	0.1	0.3	0.3	0.5	1.06633
0.03	0.1	0.5	0.3	0.5	1.06061
0.03	0.1	0.09	0.1	0.5	1.02468
0.03	0.1	0.09	0.2	0.5	1.04997
0.03	0.1	0.09	0.4	0.5	1.1018
0.03	0.1	0.09	0.5	0.5	1.12833
0.03	0.1	0.09	0.3	0.1	1.18166
0.03	0.1	0.09	0.3	0.2	1.12312
0.03	0.1	0.09	0.3	0.3	1.09901
0.03	0.1	0.09	0.3	0.4	1.08505

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