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Research Article

Optimal Bespoke CDO Design via NSGA-II

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This research work investigates the theoretical foundations and computational aspects of constructing optimal bespoke CDO structures. Due to the evolutionary nature of the CDO design process, stochastic search methods that mimic the metaphor of natural biological evolution are applied. For efficient searching the optimal solution, the nondominating sort genetic algorithm (NSGA-II) is used, which places emphasis on moving towards the true Paretooptimal region. This is an essential part of real-world credit structuring problems. The algorithm further demonstrates attractive constraint handling features among others, which is suitable for successfully solving the constrained portfolio optimisation problem. Numerical analysis is conducted on a bespoke CDO collateral portfolio constructed from constituents of the iTraxx Europe IG S5 CDS index. For comparative purposes, the default dependence structure is modelled via Gaussian and Clayton copula assumptions. This research concludes that CDO tranche returns at all levels of risk under the Clayton copula assumption performed better than the sub-optimal Gaussian assumption. It is evident that our research has provided meaningful guidance to CDO traders, for seeking significant improvement of returns over standardised CDOs tranches of similar rating.

1. Introduction

Bespoke CDOs provides tailored credit solutions to market participants. They provide both long-term strategic and tactical investors with the ability to capitalise on views at the market, sector and name levels. Investors can use these structures in various investment strategies to target the risk/return profile or hedging needs. These strategies can vary from leverage and correlation strategies to macro and relative value plays [1].

Understanding the risk/return trade-off dynamics underlying the bespoke CDO collateral portfolios is crucial when maximising the utility provided by these instruments. The single-tranche deal can be put together in a relatively short period of time. This is aided by the development of numerous advance pricing, risk management and portfolio optimisation techniques.

The most crucial tasks in putting together the bespoke CDO is choosing the underlying credits that will be included in the portfolio. Investors often express preferences on individual names, and there is likely to be credit rating constraint and industry concentration limits imposed by the investors and rating agencies [2].

Given these various investor defined requirements, the structurer is required to optimise the portfolio to achieve the best possible tranche spreads for investors. This was a complicated task, however, with the advent of faster computational pricing and portfolio optimisation algorithms, aid structurers in presenting bespoke CDO which conform to the investment parameters.

The proper implementation of the decision steps lies in the solution of the multiobjective, multiconstrained optimisation problem, where investors can choose an optimal structure that matches their risk/return profile. Optimal structures are defined by portfolios that lie on the Pareto frontier on the CDO tranche yield/portfolio risk plane.

Davidson [2] provides an interesting analogy between CDO portfolio optimisation processes and evolutionary cycles espoused by Charles Darwin. In the natural world, life adapts to suit the particulars of its environment. To adapt to a specific environment, a simple but extraordinarily powerful set of evolutionary techniques are employed—reproduction, mutation and survival of the fittest. In this way, nature explores the full range of possible structures to hone in on those that are most perfectly suited to the surrounding environment.

The creation of the CDO collateral portfolio can broadly be seen in similar ways. Given a certain set of investor and/or market constraints, such as the number of underlying credits, the notional for the credits, concentration limits and the weighted average rating factor, credit structurers need to be able to construct a portfolio that is best suited to the market environment. If the portfolio does not suit the conditions, it evolves so that only those that are "fittest," defined by having the best CDO tranche spread given the constraints, will survive. Many of the same techniques used in the natural world can be applied to this constrained portfolio optimisation problem. Evolutionary algorithms have received a lot of attention regarding their potential for solving these types of problems. They possess several characteristics that are desirable to solve real world optimisation problems up to a required level of satisfaction.

Our previous research work focused on developing a methodology to optimise credit portfolios. The Copula Marginal Expected Tail Loss (CMETL) model proposed by Jewan et al. [3], is one that minimises credit portfolio ETL subject to a constraint of achieving expected portfolio returns at least as large as an investor defined level, along with other typical constraints on weights, where both quantities are evaluated in the CMETL framework. Jewan et al. [3] have shown that ETL optimal portfolio techniques, combined with copula marginal (factor copula) distribution modelling of the portfolio risk factors can lead to significant improvements in risk-adjusted returns.

Our research work now investigates a new approach to asset allocation in credit portfolios for the determination of optimal investments in bespoke CDO tranches. Due to the complexity of the problem, advance algorithms are applied solve the constrained multiobjective optimisation problem. The nondominating sort genetic algorithm (NSGA-II) proposed by Deb et al. [4] is applied.

NSGA-II is a popular second generation multiobjective evolutionary algorithm. This algorithm places emphasis on moving towards the true Pareto-optimal region, which is essential in real world credit structuring problems. The main features of these algorithms are the implementation of a fast nondominated sorting procedure and its ability to handle

constraints without the use of penalty functions. The latter feature is essential for solving the multiobjective CDO optimisation problem.

The study uses both Gaussian and Clayton copula models to investigate the effects of different default dependence assumptions on the Pareto frontier. Various real world cases are considered, these include the constrained long-only credits and concentrated credit cases. Two objectives are used to define the CDO optimisation problem. The first is related to the portfolio risk, which is measured by the Expected-tail-loss (ETL). ETL is a convex risk measure and has attractive properties for asset allocation problems. The second objective is the CDO tranche return. This objective requires a CDO valuation model. We apply an extension of the Implied Factor model proposed by Rosen and Saunders [5]. The extension is a result of the application of the Clayton copula assumption. Rosen and Saunders [5] restricts their analysis to the Gaussian case, but use a multifactor framework.

The breakdown of the paper is as follows. The next section briefly discusses the mechanics of bespoke CDOs. It outlines the three improtant decision making steps involved in the structuring process. In section four, a robust and practical CDO valuation framework based on the application of the single-factor copula models given in Section 3, is presented. This is in conjunction with weighted Monte Carlo techniques used in options pricing. The results of the study on the impact of the different copula assumption on the market implied loss distribution is then presented. Finally the analysis on the implied credit tail characteristics under the various copula assumptions is given. Section 5 defines convex credit risk measures in a self contained manner. Sections 4 and 5 establish the theory behind the objective functions used in the CDO optimisation model. In Section 6 the generic model for multiobjective bespoke CDO optimisation is presented. The components of the NSGA-II are discussed and the algorithm outlined. This then paves the way to perform a prototype experiment on a bespoke CDO portfolio constructed from constituents of the iTraxx Europe IG S5 index. The final section highlights the improtant research findings and discusses several areas of future study.

2. Bespoke CDO Mechanics

A bespoke CDO is a popular second-generation credit product. This standalone single-tranche transaction is referred to as a bespoke because it allows the investor to customise the various deal characteristics such as the collateral composition, level of subordination, tranche thickness, and credit rating. Other features, such as substitution rights, may also play an improtant role [1]. In these transactions, only a specific protion of the credit risk is transferred, unlike the entire capital structure as in the case of standardised synthetic CDOs. Most of these transactions involve 100–200 liquid corporate CDS.

While the bespoke CDO provides great flexibility in the transaction parameters, it is crucial that investors understand the mechanics of the deal. A key feature of these transactions is the greater dialogue that exists between the parties during the structuring process, avoiding the "moral hazard" problem that existed in earlier CDO deals [6].

In a typical bespoke CDO transaction, there are three main decision steps for potential investors:

(1) *Credit Selection for the Reference portfolio*: The first step in structuring a bespoke CDO is the selection of the credits for the collateral portfolio. Investors can choose a portfolio of credits different from their current positions. They can also sell

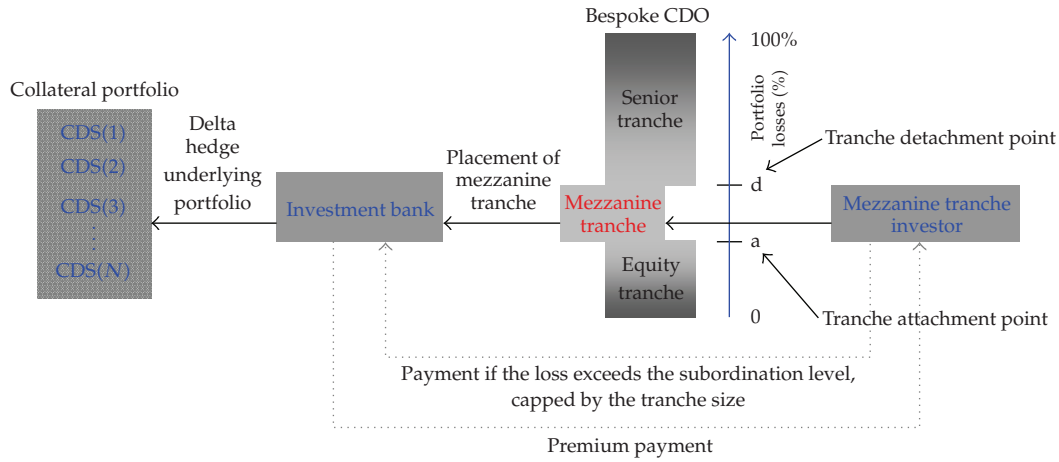

Figure 1: Placement of mezzanine tranche in a typical bespoke CDO transaction.

protection on the subset of names in their current portfolio in case they have overweight views on particular credit and/or sectors.

(2) *Defining the Subordination Level and Tranche Size*: Once a credit portfolio has been selected, investors must choose a subordination level and tranche size. These variables determine the degree of leverage and the required protection premium. Investors that are primarily concerned about the credit rating of the CDO tranche, rated by major rating agencies such as Moody's, S&P and Fitch, could choose the tranche size and subordination level so as to maximise the premium for a selected rating [1]. Usually investors choose a subordination level that will provide a total spread equal to some desired return target. This "tranching" of credit portfolio risk can provide any desired risk/return profile. By choosing the position of a tranche on a capital structure, investors can decouple their views on default risk from their views on market risk. (See Rajan et al. [1, page 203–220] for trading strategies involving bespoke CDO transactions.)

(3) *Substitution of Credits in the Reference portfolio*: The third distinctive feature of bespoke CDOs is the investors' ability to dynamically manage their investment profit and loss by substituting credits in the portfolio.

An optimal transaction would be based on a collateral portfolio that lies on the efficient frontier on the tranche return/risk plan, solution of a multiobjective optimisation problem that satisfies both trade and regulatory constraints.

A typical placement of a bespoke CDO is outlined in Figure 1.

In the above schematic, the investor goes long the credit risk in a mezzanine tranche. The challenge of issuing bespoke CDOs is the ongoing need and expense of the risk management of the tranche position. The common method of hedging these transactions is to manage the risk like an options book. Greeks similar to those related to options can be defined for CDO tranches. The distribution of risk in these transactions is not perfect, leaving dealers exposed to various first and second order risks [7].

3. Factor Copula Models for Default Dependence Modelling

Copula-based credit risk models were developed to extend the univariate credit-risk models for the individual obligors to a multivariate setting which keeps all salient features of the individual credit-risk models while incorporating a realistic dependency structure between the obligor defaults.

Copulas were first introduced by Sklar [8], in the context of probabilistic metric spaces, however their applications to finance have been very recent. The idea was first invoked within finance by Embrechths et al. [9], in connection with the inadequacies of linear correlation as a measure of dependence or association.

In credit portfolio modelling the copula approach and factor models have become an industry-wide standard to describe the asset correlation structure. Construction of mutual asset correlations between entities of the portfolio on common external factors represents a very flexible and powerful modelling tool. This modelling approach can be understood as a combination of the copula and firm-value approach. This factor approach is quite standard in credit risk modelling [10–14]. These models have been widely applied to credit derivatives modelling for essentially two reasons:

(i) factor models represent an intuitive framework and allow fast calculation of the loss distribution function; and

(ii) the full correlation matrix, which represents a challenging issue in large credit portfolios, need not be fully estimated.

Factor models can then be used to describe the dependency structure amongst credits using a so called "credit-versus-common factors" analysis rather than a pairwise analysis.

Credit risk models can be divided into two mainstreams, structural models and reduced form models. In the reduced-form methodology, the default event in these models is treated exogenously. The central idea is to model the default counting process N_c. This is a stochastic process which assumes only integer values. It literally counts default events, with $N_c(t)$ denoting the number of events that have occurred up to time t. In the case of the mth-to-default, the default time is given by the following:

$$\tau_m = \min\left\{t \in [0,T] \mid N_c(t) = m\right\}. \tag{3.1}$$

Using the standard settings for financial product pricing, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where the model lives. Let $\mathbb{Q}$ denote the risk neutral probability measure. All subsequently introduced filtrations are subsets of the filtration $\mathcal{F}$ and augmented by the zero-sets of $\mathcal{F}$ [15].

We will consider a latent factor V such that conditionally on V, the default times are independent. The factor approach makes it simple to deal with a large number of names and leads to very tractable results. We will denote by $p_t^{i|V} = \mathbb{Q}(\tau^{(i)} \le t \mid \widehat{\mathcal{H}_t} \vee \sigma(V_t))$, the conditional default probability of the name i, and by $q_t^{i|V} = \mathbb{Q}(\tau^{(i)} > t \mid \widehat{\mathcal{H}_t} \vee \sigma(V_t))$, the corresponding conditional survival probability. Conditionally on V, the joint survival function is given by the following:

$$\mathbb{Q}\left(\tau^{(1)} \ge t, \tau^{(2)} \ge t, \ldots, \tau^{(m)} \ge t \mid \widehat{\mathcal{H}_t} \vee \sigma(V_t)\right) = \prod_{i=1}^{n} \mathbb{E}\left[N_c^{(i)}(t) \mid \widehat{\mathcal{H}_t} \vee \sigma(V_t)\right] \tag{3.2}$$

for m reference entities. The filtration, $\mathscr{H}_t$ represents all available information in the economy at time t. The filtration $\mathscr{H}_t^i$ represents all available information restricted to obligor i and the background process. These filtrations enable us to model the intensity of the default process of an obligor independent of the information about the remaining obligors.

The next two subsections provide a description of the two copula models used in the study. These two models will define the default dependence structure, which is used in a Monte Carlo simulation, to derive the credit portfolio loss distribution. These distributions are then used in pricing, portfolio and risk management.

3.1. Gaussian Copula

A convenient way to take into account the default dependence structure is through the Gaussian copula model. This has become the market standard in pricing multiname credit products. In the firm-value approach a company will default when its "default-like" stochastic process, X, falls below a barrier. Define the stochastic process,

$$X_i = \rho_i V + \sqrt{1-\rho_i^2}\epsilon_i, \tag{3.3}$$

where V and ϵ_i are independent standard Gaussian variants, with $\text{Covar}(\epsilon_i, \epsilon_j) \neq 0$ for all $i \neq j$. When $\rho = 0$, this corresponds to independent default times while $\rho = 1$ is associated with the comonotonic case. X_i can be interpreted as the value of the assets of the company, and V, the general state of the economy. The default dependences come from the factor V. Unconditionally, the stochastic processes are correlated but conditionally they are independent.

The default probability of an entity i denoted by F_i can be observed from market prices of credit default swaps. Under the copula model, each X_i is mapped to $\tau^{(i)}$ using a percentile-to-percentile transformation. In a Gaussian copula model the point $X_i = x$ is transformed into $\tau^{(i)} = t$ with $\tau^{(i)} = F^{-1}(\Phi(x))$. It follows from (3.3) that the conditional default probability is given by:

$$\mathbb{Q}\left(X_i \leq x \mid V\right) = \Phi\left(\frac{x - \rho_i V}{\sqrt{1-\rho_i^2}}\right), \tag{3.4}$$

where $x = \Phi^{-1}(F_i(t))$, and $\mathbb{Q}(X \leq x) = \mathbb{Q}(\tau \leq t)$,

$$\mathbb{Q}\left(\tau^{(i)} \leq t \mid V\right) = \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i V)}{\sqrt{1-\rho_i^2}}\right). \tag{3.5}$$

We also have that:

$$\tau^{(i)} = F^{-1}\left(\Phi\left(\rho_i V + \sqrt{1-\rho_i^2}\epsilon_i\right)\right), \tag{3.6}$$

this is used when we want to simulate the default times. The Gaussian copula has no upper or lower tail dependence. Embrechths et al. [9] proves this result.

3.2. Clayton Coupla

Most of the existing copula models involve using the Gaussian copula which has symmetric upper and lower tails, without any tail dependence. This symmetry fails to capture the fact that firm failures occur in cascade in difficult times but not in better times. That is, the correlation between defaults increases in difficult times. The Clayton copula encapsulates this idea.

This "class" of Archimedean copulas was first introduced by Clayton [16] from his studies on the epidemiological chronic diseases. Clayton [16] only developed a general form (without imposing any parametric constraints) while Oakes [17] refined the copula in term of its parameterisation. Friend and Rogge [18], Laurent and Gregory [19], Schloegl and O'Kane [20], Schönbucher and Schubert [15], and Schönbucher [21] have been considering this model in a credit risk context, primarily due to the lower tail dependence which is ideal for credit risk applications.

The asset value process in a single factor Clayton copula is given by the following:

$$X_i = \left(1 - \frac{\log(\epsilon)}{V}\right)^{-1/\theta_C}, \tag{3.7}$$

where V is the systematic risk factor, a positive random variable, following a standard Gamma distribution with shape parameter $1/\theta_C$ with $\theta_C > 0$, and scale parameter equal to unity.

Using the definition of default times and the Marshall and Olkin [22] sampling algorithm, the conditional survival probability given by the following:

$$q_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta_C}\right)\right). \tag{3.8}$$

The default times are given by,

$$\tau^{(i)} = F^{-1}\left(\left(1 - \frac{\log(\epsilon)}{V}\right)^{-1/\theta_C}\right). \tag{3.9}$$

From expression (3.9), it is clear that the stochastic intensities are proprotional to V. Thus the latent variable acts as a multiplicative effect on stochastic intensities. High levels of the latent variable are associated with shorter survival default times. For this reason, V is called a "frailty."

When $\theta_C = 0$, we obtain the product copula and this implies the default times are independent. When $\theta_C \to \infty$, the Clayton copula turns out to be the upper Frëchet bound, corresponding to the case where default times are comonotonic [19]. As the parameter θ_C increases, the Clayton copula increases with respect to the supermodular order. This implies an increasing dependence in default times, and hence has some direct consequences for the pricing of CDO tranches as shown by Laurent and Gregory [19].

4. Implied Factor Copula Models for CDO Valuation

The current market standard for pricing synthetic CDOs is the single-factor Gaussian copula model introduced by Li [23]. Numerous research has shown that a single parameter model is unable to match the price of all quoted standardised CDO tranches [24–32].

It is common practice to quote an implied "correlation skew"—a different correlation which matches the price of each tranche. This assumption is analogous to the Black-Scholes implied volatility in the options market. Implied tranche correlations not only suffer from interpretation problems, but they might not be unique as in the case of mezzanine tranches, and cannot be interpolated to price bespoke tranches [33].

We define bespoke tranches in the following cases:

(i) the underlying portfolio and maturity is the same as the reference portfolio, but the tranche attachment and/or detachment points are different,

(ii) the underlying portfolio is the same as the reference portfolio but the maturities are different, or

(iii) the underlying portfolio differs from the reference portfolio.

The following subsection presents a robust and practical CDO valuation framework based on the application of the single-factor copula models presented in the previous section. The method to recover the credit loss distributions from the factor copula structure is then presented. The implied factor model is then derived. This development is in conjunction with weighted Monte Carlo techniques used in options pricing. The Gaussian model presented here is a special case of the multifactor Gaussian copula model proposed by Rosen and Saunders [5]. The application of the Clayton copula model is an extension of Rosen and Saunders [5] work.

The impact of the different copula assumptions on the loss distribution is also investigated. The credit tail characteristics are analysed. This is imperative to ensure that the copula model used has the ability to capture the default dependence between the underlying credits, and not severely underestimate the potential for extreme losses. The loss analysis is performed on a homogeneous portfolio consisting of 106 constituents of the iTraxx European IG S5 CDS index.

4.1. Synthetic CDO Pricing

The key idea behind CDOs, is the tranching of the credit risk of the underlying portfolio. A given tranche n_{tr} is defined by its attachment and detachment points $u_{n_{\text{tr}}}^{\text{lower}}$ and $u_{n_{\text{tr}}}^{\text{upper}}$ respectively. The tranche notional is given by: $s_{n_{\text{tr}}} = N_{\text{prot}}(u_{n_{\text{tr}}}^{\text{upper}} - u_{n_{\text{tr}}}^{\text{lower}})$, where N_{prot} denotes the total portfolio value.

Let $L_{\text{total}}(t)$ be the percentage cumulative loss in the portfolio value at time t. The total cumulative loss at time t is then $L_{\text{total}}(t)N_{\text{prot}}$. The loss suffered by the holders of tranche n_{tr} from origination to time t is a percentage $L_{n_{\text{tr}}}(t)$ of the portfolio notional value N_{prot}:

$$L_{n_{\text{tr}}}(t) = \min\left\{ \max\left\{L_{\text{total}}(t) - L_{n_{\text{tr}}}, 0\right\}, u_{n_{\text{tr}}}^{\text{upper}} - u_{n_{\text{tr}}}^{\text{lower}}\right\}. \tag{4.1}$$

We consider a transaction initiated at time 0, with maturity T. In a CDO contract, the tranche investor adopts a position as a protection seller. Similar to the assumption under the CDS

pricing, we assume that defaults occur at the midpoints between coupon payment dates. The value of the protection leg is then given by the following:

$$\mathbb{E}\left[\mathrm{PV}_{\mathrm{prot}}^{(n_{\mathrm{tr}})}\right] = \sum_{j=1}^{J} D\left(0, \frac{t_j - t_{j-1}}{2}\right)\left(\mathbb{E}\left[L_{n_{\mathrm{tr}}}(t_j)\right] - \mathbb{E}\left[L_{n_{\mathrm{tr}}}(t_{j-1})\right]\right), \tag{4.2}$$

where $\mathbb{E}[L_{n_{\mathrm{tr}}}(\,\cdot\,)]$ is the expectation with respect to the risk neutral measure $\mathbb{Q}$ and is calculated by a simulation model. The tranche loss profiles under each scenario are calculated and stored. The expectation is found by calculating the weighted average over all scenarios. This only applies if a weighted Monte Carlo scheme is used.

The tranche investors need to be compensated for bearing the default risk in the underlying credits. The holders of tranche n receive a periodic coupon payment. Let the coupon payment dates be denoted as $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J = T$. The predetermined frequency of the coupon payment dates is usually on a quarterly basis. The spread paid for protection on a given tranche does not vary during the life of the contract. and is usually quoted in basis points per annum. However, the tranche notional decays through the life of the contract. The outstanding tranche notional at time t is given by the following:

$$N_{\mathrm{out}}^{n_{\mathrm{tr}}}(t) = \left(u_{n_{\mathrm{tr}}}^{\mathrm{upper}} - u_{n_{\mathrm{tr}}}^{\mathrm{lower}} - \mathbb{E}\left[L_{n_{\mathrm{tr}}}(t)\right]\right)N_{\mathrm{prot}}. \tag{4.3}$$

The expected outstanding tranche notional since the last coupon date must be considered at the coupon payment dates. This amount between coupon payment dates t_j and t_{j-1} is simply the average of $N_{\mathrm{out}}^{n_{\mathrm{tr}}}(t_j)$ and $N_{\mathrm{out}}^{n_{\mathrm{tr}}}(t_{j-1})$. We assume again that defaults can only occur at the midpoint between arbitrary coupon dates. The expected outstanding tranche notional is denoted by:

$$\mathbb{E}\left[N_{\mathrm{out}}^{n_{\mathrm{tr}}}(t_j, t_{j-1})\right] = \left(u_{n_{\mathrm{tr}}}^{\mathrm{upper}} - u_{n_{\mathrm{tr}}}^{\mathrm{lower}} - \mathbb{E}\left[L_{n_{\mathrm{tr}}}(t_j)\right] + \frac{\mathbb{E}[L_{n_{\mathrm{tr}}}(t_j)] - \mathbb{E}[L_{n_{\mathrm{tr}}}(t_{j-1})]}{2}\right). \tag{4.4}$$

Using this equation we can compute the expected present value of the coupon payments:

$$\mathbb{E}\left[\mathrm{PV}_{\mathrm{prem}}^{n_{\mathrm{tr}}}\right] = \sum_{j=1}^{J} s_{n_{\mathrm{tr}}}(t_j - t_{j-1})D(0, t_j)\mathbb{E}\left[N_{\mathrm{out}}^{n_{\mathrm{tr}}}(t_j, t_{j-1})\right]. \tag{4.5}$$

This fair spread of a tranche can be computed by equating the expected present value of the protection and premium legs. The market quotation for the equity tranche is to have a fixed 500 bps spread, and the protection buyer make an upfront payment of a fixed percentage of the tranche notional. The protection seller receives the upfront fee expressed as a percentage f of the tranche notional, so that equity investors purchase the note at a discount $f(u_{\mathrm{eq}})N_{\mathrm{prot}}$. Only the premium leg is different for equity tranche investors. This is given by the following:

$$\mathbb{E}\left[\mathrm{PV}_{\mathrm{prem}}^{(\mathrm{eq})}\right] = f(u_{\mathrm{eq}})N_{\mathrm{prot}} + \sum_{j=1}^{J} s_{\mathrm{eq}}(t_j - t_{j-1})D(0, t_j)\mathbb{E}\left[N_{\mathrm{out}}^{(\mathrm{eq})}(t_j, t_{j-1})\right]. \tag{4.6}$$

4.2. Weighted Monte Carlo Techniques

Monte Carlo algorithms can be divided (somewhat arbitrarily) into two categories: uniformly weighted and nonuniformly weighted algorithms. nonuniform weights are a mechanism for improving simulation accuracy. Consider a set of M paths, generated by a simulation procedure. A nonuniformly weighted simulation is one in which the probabilities are not necessarily equal. Suppose that we assign, respectively, probabilities $p_1, p_2, \ldots, p_M$, to the different paths. The value of the security according to the nonuniform weights is,

$$\Omega_h = \sum_{m=1}^{M} p_m \Lambda_m, \tag{4.7}$$

where Λ is the payoff function.

Two features of credit risk modelling which pose a particular challenge under simulation based procedures, namely:

(1) it requires accurate estimation of low-probability events of large credit losses; and

(2) default dependence mechanisms described in the previous chapter do not immediately lend themselves to rare-event simulation techniques used in other settings.

It is for these reasons that we implement a weighted Monte Carlo simulation procedure to put emphasis on determining directly the risk neutral probabilities of the future states of the market, which will allow the accurate determination of the credit loss distribution. This is in contrast to calibration of pricing models through traditional methods.

In what follows, we introduce the general modelling framework, present the algorithm and discuss some technical implementation details. (We use a similar methodology to Rosen and Saunders [5].) This generalised framework can be applied to credit risk models that have a factor structure.

A weighted Monte Carlo method can be used to find an implied risk-neutral distribution of the systematic factors, assuming the specification of a credit risk model with a specified set of parameter values. According to Rosen and Saunders [5] the justification for this approach follows working within a conditional independence framework, where obligor defaults are independent, conditional on a given systematic risk factor.

The methodology to obtain the implied credit loss distribution is summarised by the following steps.

(i) *Latent factor scenarios*. Define M scenarios on the systematic factor V.

(ii) *Conditional portfolio loss distribution*. Calculate the conditional portfolio loss profile for scenario m.

(iii) *Conditional tranche values*: Infer the tranche loss distributions conditional on scenario m.

(iv) *Implied scenario probabilities (optimisation problem)*. Given a set of scenario probabilities p_m; tranche values are given as expectations over all scenarios of the conditional values. We solve the resulting constrained inverse problem to find a set of implied scenario probabilities p_m.

(v) *Implied credit loss distribution*-Calculate the aggregate credit loss at each time step given the implied scenario probabilities.

The first three steps above have been discussed in detail in the previous sections of this chapter. The focus now is placed on the optimisation problem for implying the scenario probabilities.

4.2.1. Mathematical Formulation

Let $h : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ be a strictly convex function and let $w_{i,M}$ denote the weight of the ith path of M paths. Now consider the optimisation problem:

$$\min_{w_{1,M}, w_{2,M}, \ldots, w_{M,M}} \sum_{i=1}^{M} h(w_{i,M}) \quad \text{subject to}$$

$$\frac{1}{M} \sum_{i=1}^{M} w_{i,M} = 1, \tag{4.8}$$

$$\frac{1}{M} \sum_{i=1}^{M} w_{i,M} \mathbf{G}_i = c_{\mathbf{G}},$$

for some fixed $c_{\mathbf{G}} \in \mathbb{R}^N$. The objective function is strictly convex and the constraints are linear, so if a feasible solution exist, with a finite objective function value, then there is a unique optimal solution $w_{1,M}, w_{2,M}, \ldots, w_{M,M}$. This optimum defines the weighted Monte Carlo estimator,

$$\widehat{\Psi}_{\mathrm{LCV}} = \sum_{i=1}^{M} w_{i,M} \Psi_i. \tag{4.9}$$

The weights derived from (4.8) can be made more explicit by introducing the Lagragian. The strict convexity of the objective function is not a sufficient condition to guarantee a unique solution.

The classical approach to solving constrained optimisation problems is the method of Lagrange multipliers. This approach transforms the constrained optimisation problem into an unconstrained one, thereby allowing the use of the unconstrained optimisation techniques.

4.2.2. Objective Functions

Taking the objective to be a symmetric separable convex function gives the optimal probabilities p. This is an interpreted as the most uniform probabilities satisfying the constraints, though different objective functions imply different measures of uniformity. A common choice for the fitness measure is *entropy*. This is a particularity interesting and in some respects a convenient objective. The principle of maximum entropy gives a method of generating a probability distribution from a limited amount of information. It is a relatively well used principle for the construction of probabilistic models.

In this setting,

$$h = \sum_{w^*=1}^{M} p_{w^*} \log p_{w^*}. \tag{4.10}$$

This case is put forward in Avellaneda et al. [34], where a Bayesian interpretation is given. The usual convention of $0 \log 0$ is followed. We now provide a problem specific description of the optimisation problem using the principle of maximum entropy.

4.2.3. Constraint Description

For the bespoke CDO pricing problem, the implied scenario probabilities must satisfy the following constraints:

(i) the sum of all scenario probabilities must equal to one;

(ii) the probabilities should be positive: $p_i \geq 0$ for all $i \in \{0, 1, \ldots, M\}$; and bounded by 1;

(iii) we match the individual CDS spreads (i.e., marginal default probabilities for each name)

$$\sum_{w^*=1}^{M} p_{w^*} p_{z^*}^{t|V_{w^*}} = p_{z^*}^{t|V} \quad \forall z^* \in \{1, 2, \ldots, N\}; \tag{4.11}$$

(iv) the current market prices of standard CDO tranches are matched given by

$$\sum_{w^*=1}^{M} p_{w^*} \mathrm{PV}_{\mathrm{prot}}^{(n_{\mathrm{tr}})} = \sum_{w^*=1}^{M} p_{w^*} \mathrm{PV}_{\mathrm{prem}}^{(n_{\mathrm{tr}})} \quad \forall n_{\mathrm{tr}} \in \{1, 2, \ldots\} \tag{4.12}$$

tranches.

Because the number of controls in the problem is typically smaller than the number of replications, these constraints do not determine the probabilities [35]. We choose a particular set of probabilities by selecting the maximum entropy measure as the objective. The problem specific optimisation problem is defined by,

$$\begin{gathered}
\max_{\mathbf{p} \in \mathbb{R}^M} \sum_{w^*=1}^{M} p_{w^*} \log p_{w^*} \quad \text{subject to} \\
\sum_{w^*=1}^{M} p_{w^*} p_{z^*}^{t|V_{w^*}} = p_{z^*}^{t|V} \quad \forall z^* \in \{1, 2, \ldots, N\}, \\
p_{w^*} \geq 0 \quad \forall w^* \in \{0, 1, \ldots, M\}, \\
\sum_{w^*=1}^{M} p_{w^*} = 1.
\end{gathered} \tag{4.13}$$

The CDO tranche prices constraint will only hold if the bespoke and index collateral portfolios have the same underlying, while the constraint on default probabilities will always hold.

4.2.4. Augmented Lagrangian Methods

The augmented Lagragian method seeks the solution by replacing the original constrained problem with a sequence of unconstrained subproblems in which the objective function is formed by the original objective of the constrained optimisation plus additional "penalty" terms. These terms are made up of constraint functions multiplied by a positive coefficient.

For the problem at hand, the augmented Lagrangian function is given by,

$$L_M^*(x;s;\lambda_M;\mu_M) = \sum_{m=1}^{M} p_m \log p_m + \lambda_M^{(1)}\left(1-\sum_{i=1}^{M} p_{i,M}\right) + \lambda_M^{(2)}\sum_{k=1}^{N}\left(p_k^{t|V} - \sum_{i=1}^{M} p_{i,M} p_k^{t|V_m}\right) + \frac{1}{2\mu_M}\left(1-\sum_{i=1}^{M} p_{i,M}\right)^2 + \frac{1}{2\mu_M} + \left(\sum_{k=1}^{N}\left(p_k^{t|V} - \sum_{i=1}^{M} p_{i,M} p_k^{t|V_m}\right)\right)^2. \tag{4.14}$$

4.2.5. Implementation Issues

In practice, a significant proprotion of computation time is not spent solving the optimisation problem itself, but rather computing the coefficients in the linear constraints. The marginal default probability constraints require the evaluation of the conditional default probability given in the previous chapter for each name under each scenario.

One has the option to only match the cumulative implied default probability to the end of the lifetime of the bespoke CDO or perhaps at selected times only. The advantage of dropping constraints is twofold. Firstly it reduces the computational burden by reducing the number of coefficients that need to be computed, thus leading to a faster pricing algorithm. Secondly, it loosens the conditions required of the probabilities $\mathbf{p}$, thus resulting in factor implied distributions with superior "quality," as given by higher values for the fitness function [5]. Matlab is used for implementing the model.

4.3. Interpreting the Implied Distributions

The valuation of CDOs depends on the portfolio loss distribution. For the pricing of a CDO or CDO^2 it is sufficient to know the portfolio loss distributions over different time horizons. The implied credit loss distributions should be considered relative to the prior models, before deriving efficient frontiers for the bespoke portfolios. This is crucial as deviations from the model will result in a sub-optimal asset allocation strategy.

Figure 2 shows the implied and model distribution of default losses for the benchmark portfolio under the Gaussian copula assumption. Kernel smoothing was applied to the density results. This approach is a nonparametric way of estimating the probability density function of a random variable. This density estimation technique makes it possible to extrapolate the data to the entire population.

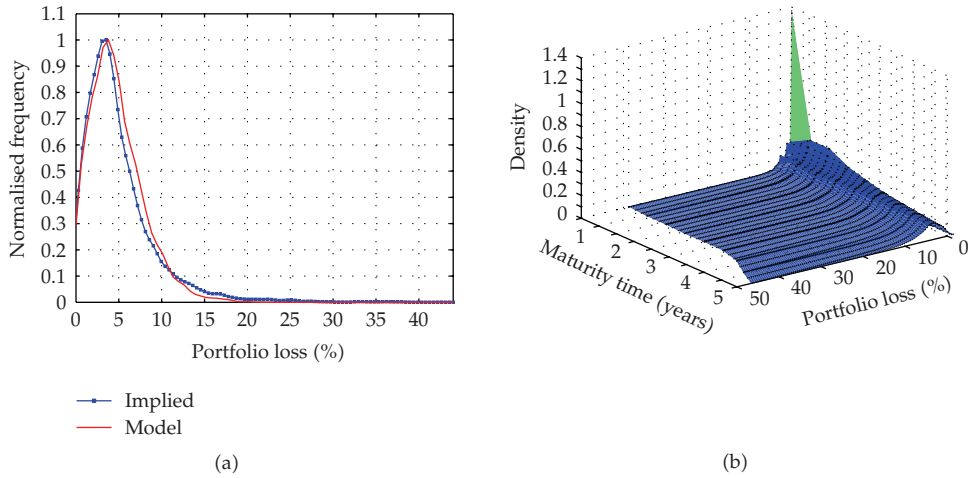

Figure 2: The comparison of the implied loss density under Gaussian copula assumption: (a) shows the deviation of the model density from the implied density at the 5 year horizon, (b) displays the implied credit loss density surface up to the 5 year horizon.

The first thing to note is the typical shape of credit loss distributions. Due to the common dependence on the factor V, defaults are correlated. This association gives rise to a portfolio credit loss density that is right-skewed and has a right-hand tail. The loss surface as a whole is widening and flattening, with an increasing expected portfolio loss.

The Clayton copula model displays similar deviations. This is shown in Figure 3. The Clayton assumption will still under-estimate small losses, but this effect is less pronounced than the Gaussian case. The maximum portfolio loss obtained over a 5 year horizon is 53.7% as compared to the 44.3% in the Gaussian case. The loss surface as a whole is widening and flattening with an increasing expected portfolio loss. The maximum loss increases from 44.2% over a 6 month horizon to 53.7% over five years. The Clayton copula model still does not perfectly capture the market dynamics, but is an improvement over the Gaussian case.

Similar sentiments on the weakness of Gaussian copula are shared by Li and Liang [36].

Figure 4 uses a logarithmic scale for the probabilities to show the tail effects more clearly.

The probabilities decrease very quickly under the Gaussian and Clayton copula assumptions. The effect of thicker tails under the Clayton copula can easily be seen to dominate the Gaussian copula. In the pricing of the super senior tranche, the Clayton model exhibits higher expected losses due to the excess mass concentrated in the tails of the distribution, the tranche spread will be higher under this model than the Gaussian case. These deviations in the implied distribution under different distributional assumptions will filter through to the resulting efficient frontiers. Due to the higher tail probabilities the Clayton ETL efficient frontiers will be significantly different from frontiers resulting from the Gaussian assumption. This feature will be shown in the subsequent sections.

The results so far also have a practical edge for credit risk management. The likelihood of extreme credit losses is increased under the Clayton copula assumption. This is due to the lower tail dependency exhibited by this copula function.

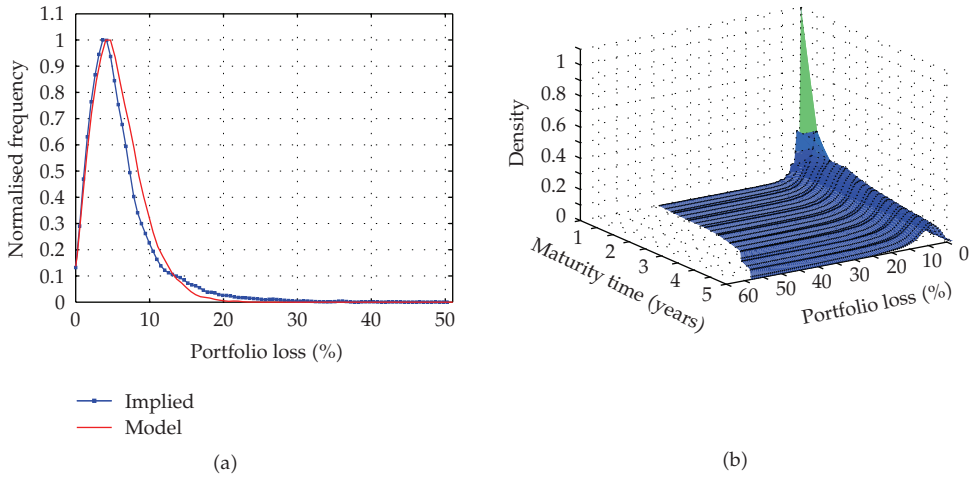

Figure 3: The comparison of the implied loss density under Clayton copula assumption: (a) shows the deviation of the model density from the implied density at the 5 year horizon, (b) displays the implied credit loss density surface up to the 5 year horizon.

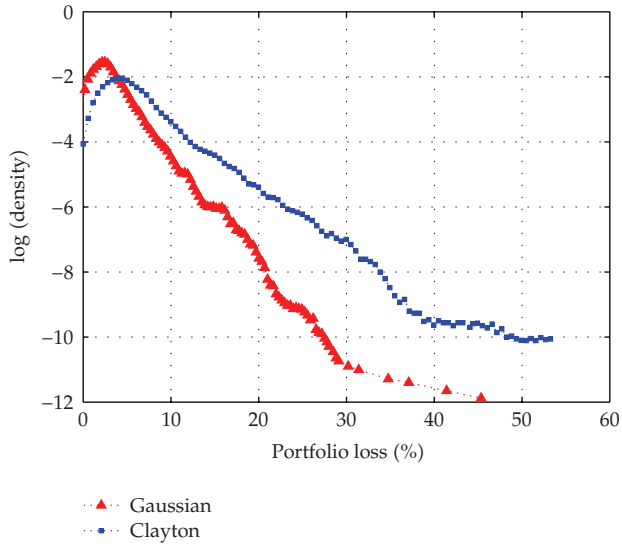

Figure 4: The comparison of implied tail probabilities under different copula assumptions.

5. Risk Characterisation of Credit portfolios

Comparison of uncertainty in outcomes is central to investor preferences. If the outcomes have a probabilistic description, a wealth of concepts and techniques from probability theory can be applied. The main objective in the following section is to present a review of the fundamental work by Artzner et al. [37] and Föllmer and Schied [38] from an optimisation point of view, since a credit risk measure will represent one of the objectives in the CDO optimisation problem.

Let R denote the set of random variables defined on the probability space $(\Omega_p, \mathcal{F}_p, \mathbb{Q}_p)$. We define Ω_p as a set of finitely many possible scenarios for the portfolio p. Financial risks are represented by a convex cone $\mathcal{M} \subseteq (\Omega_p, \mathcal{F}_p, \mathbb{Q}_p)$ of random variables. Any random variable L_c in this set will be interpreted as a possible loss of some credit portfolio over a given time horizon. The following provides a definition of a convex cone.

Definition 5.1 (convex cone). $\mathcal{M}$ is a convex cone if,

(1) $L_c^{(1)} \in \mathcal{M}$ and $L_c^{(2)} \in \mathcal{M}$ implies that $L_c^{(1)} + L_c^{(2)} \in \mathcal{M}$; and

(2) $\lambda^* L_c \in \mathcal{M}$ for every $\lambda^* \geq 0$.

Definition 5.2 (measures of risk). Given some convex cone, $\mathcal{M}$ of random variables, a measure of risk Θ with domain $\mathcal{M}$ is a mapping:

$$\Theta : \mathcal{M} \longrightarrow \mathbb{R}. \tag{5.1}$$

From an economic perspective, $\Theta(L_c)$, can be regarded as the capital buffer that should be set aside for adverse market movements. In order to measure and control the associated risk Artzner et al. [37] introduced an axiomatic framework of coherent risk measures which were recently "generalised" by Föllmer and Schied [38] to convex risk measures.

Definition 5.3 (Convex risk measures). A mapping $\Theta : \mathcal{M} \to \mathbb{R}$ is called a *convex risk measure*, if and only if it is

(1) *convex* for every $L_c^{(1)}$ and $L_c^{(2)} \in \mathcal{M}$, one has $\Theta(\Lambda L_c^{(1)} + (1-\Lambda)L_c^{(2)}) \leq \Lambda\Theta(L_c^{(1)}) + (1-\Lambda)\Theta(L_c^{(2)})$ for some $\Lambda \in \mathbb{R}$;

(2) *monotone* for every $L_c^{(1)}$ and $L_c^{(2)} \in \mathcal{M}$ with $L_c^{(1)} \leq L_c^{(2)}$, one has $\Theta(L_c^{(1)}) \leq \Theta(L_c^{(2)})$; and

(3) *translation invariant* if a_* is a constant then $\Theta(L_c + a_*\mathbf{1}) = -a_* + \Theta(L_c)$, where $\mathbf{1}$ denotes the unit vector.

By adding positive homogeneity to these properties, one obtains the following definition.

Definition 5.4 (coherent risk measures). A convex risk measure Θ is called *coherent*, if in addition it is,

Positively homogeneous if $\Lambda \geq 0$ then $\Theta(\Lambda L_c) = \Lambda\Theta(L_c)$ holds.

Denote the credit loss distribution of L_c by $F_{L_c}(l_c) = \mathbb{Q}(L_c \leq l_c)$. In the analysis we are concerned solely with two risk measures which are based on the loss distribution F_{L_c}, namely VaR and ETL. We now recall the definition of these risk measures.

Definition 5.5 (value-at-risk (VaR)). Given some confidence level $\beta_* \in (0,1)$, the *Value-at-Risk* (VaR) of the credit portfolio at the confidence level β_* is given by the smallest number l_c such that the probability that the loss L_c exceeds l_c is no larger than $(1-\beta_*)$. Formally,

$$\text{VaR} = \inf\left(l_c \in \mathbb{R} : \mathbb{Q}\left(L_c > l\right) \leq 1 - \beta_*\right). \tag{5.2}$$

This definition of VaR coincides with the definition of an β_*-quantile of the distribution of L_c in terms of a generalised inverse of the distribution function F_{L_c}. We observe this coincidence by noting,

$$\begin{aligned} \text{VaR} &= \inf\left(l_c \in \mathbb{R} : 1 - F_{L_c}(l_c) \leq 1 - \beta_*\right) \\ &= \inf\left(l_c \in \mathbb{R} : F_{L_c}(l_c) \geq \beta_*\right). \end{aligned} \tag{5.3}$$

For a random variable Y we will denote the β_*-quantile of the distribution by $q_{\beta_*}(F_Y)$, and write $\text{VaR}_{\beta_*}(Y)$ when we wish to stress that the quantile should be interpreted as a VaR number. A simple definition of ETL which suffices for continuous loss distributions is as follows.

Definition 5.6 (expected-tail-loss (ETL)). Consider a loss L with continuous $F_L df$ satisfying

$$\int_{\mathbb{R}} |l| dF_{L_c}(l_c) < \infty. \tag{5.4}$$

Then the *Expected-Tail-Loss* at confidence level $\alpha \in (0,1)$, is defined to be,

$$\text{ETL}_{\beta_*} = \mathbb{E}\left[L_c \mid L_c \geq \text{VaR}_{\beta_*}\right] = \frac{\mathbb{E}[L_c; L_c \geq \text{VaR}_{\beta_*}(L_c)]}{\mathbb{Q}[L_c \geq \text{VaR}_{\beta_*}(L_c)]}. \tag{5.5}$$

6. Optimal Bespoke CDO Design

Credit portfolio optimisation plays a critical role in determining bespoke CDO strategies for investors. The most crucial tasks in putting together bespoke CDOs is choosing the underlying credits that will be included in the portfolio. Usually investors often express preferences on individual names to which they willing to have the exposure, while there are likely to be credit rating constraints and industry/geographical concentration limits imposed by rating agencies and/or investors.

Given these various requirements, it is up to the credit structurer to optimise the portfolio and achieve the best possible tranche spreads for investors. In the following analysis, we focus on the asset allocation rather than credit selection strategy, which remains a primary modelling challenge for credit structurers.

Davidson [2] provides an interesting analogy between bespoke CDO optimisation and Darwin's evolutionary cycles. In the natural world, life adapts to suit the particulars of its environment. The adaptation to a specific environment is possible due to the application of powerful set evolutionary techniques—reproduction, mutation and survival of the fittest. Nature then explores the full range of possible structures to hone in on those that are most perfectly suited to the surroundings.

Creating a portfolio for a CDO can broadly be seen in similar ways. Given a certain set of investor defined, constraintsstructurers need to be able to construct a credit portfolio that is best suited to the market environment. Added to these constraints are market constraints such as trade lot restrictions and liquidity and availability of underlying credits. If the portfolio does not suit these conditions, it evolves so that only those with the best fit (highest tranche spreads) will survive. Many of the same techniques used in the natural world can be applied to this structuring process.

Evolutionary computation methods are exploited to allow for a generalisation of the underlying problem structure and to solve the resulting optimisation problems, numerically in a systematic way. The next section will briefly discuss some of the basic concepts of multiobjective optimisation and outline the NSGA-II algorithm used for solving the challenging CDO optimisation problem. The CDO optimisation model is then outlined before conducting a prototype experiment on the test portfolio constructed from the constituents of the iTraxx Europe IG S5 index.

6.1. Multiobjective Optimisation

Many real-world problems involve simultaneous optimisation of several incommensurable and often competing objectives. In single-objective optimisation the solution is usually clearly defined, this does not hold for multiobjective optimisation problems. We define formally, the multiobjective optimisation problem to define other improtant concepts used in this chapter. All problems are assumed to be minimisation problems unless otherwise specified To avoid inserting the same reference every few lines, note that the terms and definitions are taken from Zitzler [39], and adapted to CDO optimisation problem.

Definition 6.1 (multiobjective optimisation problem). A general *multiobjective optimisation Problem* (MOP) includes a set of n_* parameters (decision variables), a set of k_1 objective functions, and a set of k_2 constraints. Objective functions and constraints are functions of the decision variables. The optimisation goal is to obtain,

$$\begin{aligned} \min \mathbf{y} = \mathbf{f}(\mathbf{x}) &= \left(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{k_1}(\mathbf{x})\right) \quad \text{subject to,} \\ e(\mathbf{x}) &= \left(e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_{k_2}(\mathbf{x})\right) \leq 0, \\ \mathbf{x} &= \left(x_1, x_2, \ldots, x_n\right) \in \mathbf{X}, \\ \mathbf{y} &= \left(y_1, y_2, \ldots, y_{k_1}\right) \in \mathbf{Y}, \end{aligned} \tag{6.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the decision and objective vectors, respectively, whilst $\mathbf{X}$ and $\mathbf{Y}$ denote the decision and objective spaces, respectively. The constraints $e(\mathbf{x}) \leq \mathbf{0}$ determine the set of feasible solutions.

Without loss of generality, a minimisation problem is assumed here. For maximisation or mixed maximisation/minimisation problems the definitions are similar.

Definition 6.2 (vector relationship). For any two vectors $\mathbf{r}$ and $\mathbf{r}^*$

$$\begin{aligned} \mathbf{r} = \mathbf{r}^* \quad &\text{iff} \quad r_i = r_i^* \ \forall i \in \{1, 2, \ldots, k_1\}, \\ \mathbf{r} \lneq \mathbf{r}^* \quad &\text{iff} \quad r_i \leq r_i^* \ \forall i \in \{1, 2, \ldots, k_1\}, \\ \mathbf{r} < \mathbf{r}^* \quad &\text{iff} \quad \mathbf{r} \leq \mathbf{r}^* \wedge \mathbf{r} \leq \mathbf{r}^*. \end{aligned} \tag{6.2}$$

The relations $\geq$ and $>$ are similarly defined.

Although the concepts and terminology of Pareto optimality are frequently invoked, most often they are erroneously used in literature. We now define this set of concepts to ensure understanding and consistency.

Definition 6.3 (Pareto dominance). For any two vectors $\mathbf{r}$ and $\mathbf{r}*$

$$\begin{aligned}
&\mathbf{r} \prec \mathbf{r}^* \quad \text{iff} \quad \mathbf{f}(\mathbf{r}) \leq \mathbf{f}(\mathbf{r}^*) \text{ then } \mathbf{r} \text{ dominates } r^*,\\
&\mathbf{r} \not\preceq r^* \quad \text{iff} \quad \mathbf{f}(\mathbf{r}) \leq \mathbf{f}(\mathbf{r}^*) \text{ then } r \text{ weakly dominates } r^*,\\
&\mathbf{r} \sim \mathbf{r}^* \quad \text{iff} \quad \mathbf{f}(\mathbf{r}) \not\leq \mathbf{f}(\mathbf{r}^*) \wedge \mathbf{f}(\mathbf{r}^*) \not\leq \mathbf{f}(\mathbf{r}) \text{ then } \mathbf{r} \text{ is indifferent to } \mathbf{r}^*.
\end{aligned} \tag{6.3}$$

The definitions for a maximisation problems ($\succ, \not\succeq, \sim$) are analogous.

Definition 6.4 (Pareto optimality). A decision vector $\mathbf{x} \in \mathbf{X}_f$ is said to be nondominated regarding a set $\mathbf{K} \subset \mathbf{X}_f$ iff

$$\nexists\, \mathbf{k} \in \mathbf{K} : \mathbf{k} \prec \mathbf{K}. \tag{6.4}$$

If it is clear within the context which set $\mathbf{K}$ is meant, it is simply left out. Moreover, $\mathbf{x}$ is said to be Pareto optimal iff $\mathbf{x}$ is nondominated regarding $\mathbf{X}_f$.

The entirety of all Pareto-optimal solutions is termed the *Pareto-optimal set*; the corresponding objective vectors form the *Pareto-optimal frontier* or surface.

Definition 6.5 (nondominated sets and frontiers). Let $\mathbf{K} \subset \mathbf{X}_f$. The function $\mathcal{N}_d(\mathbf{K})$ gives the set of nondominated decision vectors in $\mathbf{K}$:

$$\mathcal{N}_d(\mathbf{K}) = \{\mathbf{k} \in \mathbf{K} \mid \mathbf{k} \text{ is nondominating regarding } \mathbf{K}\}. \tag{6.5}$$

The set $\mathcal{N}_d(\mathbf{K})$ is the nondominated set regarding $\mathbf{K}$, the corresponding set of objective vectors $\mathbf{f}(\mathcal{N}_d(\mathbf{K}))$ is the nondominated front regarding $\mathbf{K}$. Furthermore, the set $\mathbf{X}_{\mathcal{N}_d} = \mathcal{N}_d(\mathbf{X}_f)$ is called the *Pareto-optimal set* and the set $\mathbf{Y}_{\mathcal{N}_d} = \mathbf{f}(\mathbf{X}_f)$, is denoted as the *Pareto-optimal frontier*.

6.2. Nondominating Sort Genetic Algorithm (NSGA-II)

The second generation NSGA-II is a fast and elitist multiobjective evolutionary algorithm. The main features are (http://www.kxcad.net/ESTECO/modeFRONTIER320/html/userman/ch07s01s10.html).

(i) *Implementation of a fast nondominated sorting procedure*: Sorting of the individuals of a given population is according to the level of nondomination. Generally, nondominated sorting algorithms are computationally expensive for large population sizes however, the adopted solution performs a clever sorting strategy.

(ii) *Implementation of elitism for multiobjective search*: Using an elitism-preserving approach introduces storing all nondominated solutions discovered so far, beginning from the initial population. Elitism enhances the convergence properties towards the true Pareto-optimal set.

(iii) *Adopting a parameter-less diversity preservation mechanism*: Diversity and spread of solutions is guaranteed without use of sharing parameters, since NSGA-II adopts a suitable parameter-less niching approach. This niche is accomplished by

the crowding distance measure, which estimates the density of solutions in the objective space, and the crowded comparison operator, which guides the selection process towards a uniformly spread Pareto frontier.

(iv) *Constraint handling method does not make use of penalty parameters*: The algorithm implements a modified definition of dominance in order to solve constrained multiobjective problems efficiently.

(v) *Real-coded and binary-coded design variables*: A new feature is the application of the genetic algorithms in the field of continuous variables.

6.2.1. Representation of Individuals

The first stage of building the EA is to link the "real world" to the "EA world." This linking involves setting up a bridge between the original problem context and the problem-solving solving space, where evolution takes place. The objects forming possible solutions within the original problem context are referred to as a *phenotype*, while their encoding are called *genotypes* or more commonly *chromosomes*.

The mapping from phenotype space to the genotype space is termed *encoding*. The inverse mapping is termed *decoding*.

Choosing an appropriate representation for the problem being solved is improtant in the design of a successful EA. Often it comes down to good knowledge of the application domain [40].

Many different encoding methods have been proposed and used in EA development. Few frequently applied representations are: binary, integer and real valued representation. Real-valued or Floating-point representation is often the most sensible way to represent a candidate solution to of a problem. This approach is appropriate when the values that we want to represent as genes, originate from a continuous distribution.

The solutions to the proposed CDO optimisation models are real-valued. This study opted to use the real-valued encoding, for the sake of operational simplicity. The genes of a chromosome are real numbers between 0 and 1, which represents the weights invested in the different CDS contracts. However, the summation of these weights might not be 1 in the initialisation stage or after genetic operations. To overcome this problem, the weights are normalised as follows:

$$x_i' = \frac{x_i}{\sum_{i=1}^{N} x_i}. \tag{6.6}$$

6.2.2. Evaluation Function

The role of the *evaluation function* is to represent the requirements to which the population should adapt. This role forms the basis of selection. Technically, the function assigns a quality measure to the genotypes. Typically the function is composed from a quality measure in the phenotype space and the inverse representation [40]. In the evolutionary context the function is usually referred to as the fitness function.

In the bespoke CDO optimisation problem, we introduce two objective to the problem. These objectives are namely the CDO tranche return, and the portfolio tail risk measured by ETL. Various constraints are then introduced to study the dynamics of the Pareto frontier under various conditions.

6.2.3. Population Models and Diversity Preservation

Unlike operators, which operate on individuals, selection operators like parent and survivor selection work at population level. In most EA applications, the population size is constant and does not change during the evolutionary search. The *diversity* of the population is a measure of the number of different solutions present.

In NSGA-I, the well-known sharing function approach was used, which was found to maintain sustainable diversity in a population [4]. There are however two difficulties with this sharing function. According to Deb et al. [4], the performance of the sharing function method in maintaining a spread of solutions depends largely on the value of σ_s, and since each solution must be compared with all other solutions in the population, the overall complexity of the sharing function approach is $\mathbf{O}(N_{\text{pop}}^2)$.

Deb et al. [4] replace the sharing function approach with a crowded-comparison approach in NSGA-II. This eliminates both the above difficulties to some extent. This approach does not require any user-defined parameter for maintaining diversity among population members, and also has a better computational complexity.

To describe this approach, we first define a metric for density estimation and then present the description of the crowded-comparison operator.

6.2.4. Crowded Comparison Operator

The density of solutions surrounding a particular solution in the population must firstly be estimated by calculating the average distance of two points on either side of this point, along each of the objectives.

Once the nondominated sort is complete, the crowding distance is assigned. Individuals are selected based on rank and crowding distance. The crowding-distance computation requires sorting the population according to each objective function value in ascending order of magnitude. The boundary solutions (solutions with smallest and largest function values) for each objective function are assigned an infinite distance value. All other intermediate solutions are assigned a distance value equal to the absolute normalised difference in the function values of two adjacent solutions. This calculation is continued with other objective functions. The overall crowding-distance value is calculated as the sum of individual distance values corresponding to each objective. Each objective function is normalised before calculating the crowding distance. (See Deb et al. [4] for more details regarding the algorithm.)

The complexity of this procedure is governed by the sorting algorithm and has computational complexity of $\mathbf{O}(M_* N_{\text{pop}} \log N_{\text{pop}})$, where there are M_* independent sorting of at most N_{pop} individuals, when all population members are in one front $\mathcal{O}$.

After all population members in the set $\mathcal{O}$ are assigned a distance metric, we can compare two solutions for their extent of proximity with other solutions. A solution with a smaller value of this distance measure is, in some sense, more crowded by other solutions.

6.2.5. Nondomination Sorting Approach

In order to identify solutions in the NSGA of the first nondominated front, each solution is compared with every other solution in the population to find if it is dominated (We summarise the algorithm discussed in Deb et al. [4].). This procedure requires $\mathbf{O}(k_1 N_{\text{pop}})$

comparisons for each solution, where k_1 is the number of objectives. This process is continued to find all members of the first nondominated level in the population, the total complexity is $\mathbf{O}(k_1 N_{\text{pop}}^2)$ computations. At this stage, all individuals in the first nondominated front are found. In order to find the individuals in the next nondominated front, the solutions of the first front are discounted temporarily and the above procedure is repeated. In the worst case, the task of finding the second front also requires $\mathbf{O}(k_1 N_{\text{pop}}^2)$ computations, particularly when number of solutions belong to the second and higher nondominated levels. This argument is true for finding third and higher levels of nondomination.

The worst case is when there are N_{pop} fronts and there exists only one solution in each front. This requires an overall $\mathbf{O}(k_1 N_{\text{pop}}^3)$ computations. For each solution we calculate two entities:

(1) the domination count n_q, the number of solutions which dominate the solution; and

(2) the set of solutions $\mathbf{H}_q$ that the solution dominates. This requires $\mathbf{O}(k_1 N_{\text{pop}}^2)$ comparisons.

Solutions in the first nondominated front will have their domination count as zero. For each solution with $n_q = 0$, each member $q \in \mathbf{H}_q$ is visited and reduce its domination count by one. By doing so, these members belong to the second nondominated front. This process continues until all fronts are identified. For each solution in the second or higher level of nondomination, the domination count can be at most $N_{\text{pop}} - 1$. Since there are at most such $N_{\text{pop}} - 1$ solutions, the total complexity of the procedure is $\mathbf{O}(N_{\text{pop}}^2)$. Thus, the overall complexity of the procedure is $\mathbf{O}(k_1 N_{\text{pop}}^3)$.

In NSGA-II the complexity reduction is due to the realisation that the body of the first inner loop (for each $p_i^* \in \mathcal{J}_i$) is executed exactly N_{pop} times as each individual can be the member of at most one front and the second inner loop (for each $q \in \mathbf{H}_p$) can be executed at maximum $N_{\text{pop}} - 1$ times for results in the overall computations. It is improtant to note that although the time complexity has reduced to $\mathbf{O}(k_1 N_{\text{pop}}^2)$, the storage requirement has increased to $\mathbf{O}(N_{\text{pop}}^2)$.

6.2.6. Parent Selection Mechanism

The selection operator determines, which individuals are chosen for mating and how many offspring each selected individual produces. Once the individuals are sorted based on nondomination with crowding distance assigned, the selection is carried out using a crowded comparison operator described above. The comparison is carried out as below based on

(1) nondomination rank p_{rank}^*, that is, individuals in FRONT_i will have their rank $p_{\text{rank}}^* = i$; and

(2) crowding distance $\text{FRONT}_i(d_j)$.

The selection of individuals is through a binary tournament selection with the crowed-comparison operator. In *tournament selection* the individuals are chosen randomly from the population and the best individual from this group is selected as parent. This process is repeated as often as individuals must be chosen. These selected parents produce uniform at random offspring. The parameter for tournament selection is the tournament size. This takes values ranging from 2 to the number of individuals in population [41].

6.2.7. Mutation Operation

Mutation causes individuals to be randomly altered. These variations are mostly small. They will be applied to the variables of the individuals with a low probability. Offspring are mutated after being created by recombination. Mutation of real variables means, that randomly created values are added to the variables with a low mutation probability. The probability of mutating a variable is inversely proprotional to the number of variables (dimensions). The more dimensions one individual has, the smaller is the mutation probability. Different papers reproted results for the optimal mutation rate [41].

The mutation step size is usually difficult to choose. The optimal step-size depends on the problem considered and may even vary during the optimisation process. Small mutation steps are often successful, especially when the individual is already well adapted. However, large mutation steps can produce good results with a faster convergence rate. According Pohleim [41] good mutation operators should often produce small step-sizes with a high probability and large step-sizes with a low probability.

In the NSGA-II a polynomial mutation operator is used. This operator is defined by the following:

$$\Psi_i^{\text{offspring}} = \Psi_i^{\text{parent}} + \left(\Psi_i^{\text{up}} + \Psi_i^{\text{low}}\right)\delta_i, \tag{6.7}$$

where $\Psi_i^{\text{offspring}}$ is the child, and Ψ_i^{parent} is the parent, with Ψ_i^{up} and Ψ_i^{low} being the upper and lower bounds of the parent component. δ_i is a small variation which is calculated from a polynomial distribution given below.

$$\delta_i = \begin{cases} \left(2r_i\right)^{1/(\eta_m+1)} - 1 & r_i < 0.5 \\ 1 - \left(2\left(1 - r_i\right)\right)^{1/(\eta_m+1)} - 1 & r_i \geq 0.5, \end{cases} \tag{6.8}$$

where $r_i \sim \mathcal{U}(0,1)$ and η_m is mutation distribution index.

6.2.8. Recombination Operation

The recombination operator produces new individuals in combining the information contained in two or more parents in the mating population. This mating is done by combining the variable values of the parents. Depending on the representation of the variables different methods must be used.

Using real-value representation, the *Simulated Binary Crossover* operator is used for recombination and *polynomial mutation* is used for mutating the offspring population [42].

Simulated binary crossover simulates the binary crossover observed in nature and is give by the following.

$$\begin{aligned} \Psi_{1,i}^{\text{offspring}} &= \frac{1}{2}\left(\left(1 - \beta_k\right)\Psi_{1,k}^{\text{Parent}} + \left(1 + \beta_k\right)\Psi_{2,k}^{\text{Parent}}\right), \\ \Psi_{2,i}^{\text{offspring}} &= \frac{1}{2}\left(\left(1 + \beta_k\right)\Psi_{1,k}^{\text{Parent}} + \left(1 - \beta_k\right)\Psi_{2,k}^{\text{Parent}}\right), \end{aligned} \tag{6.9}$$

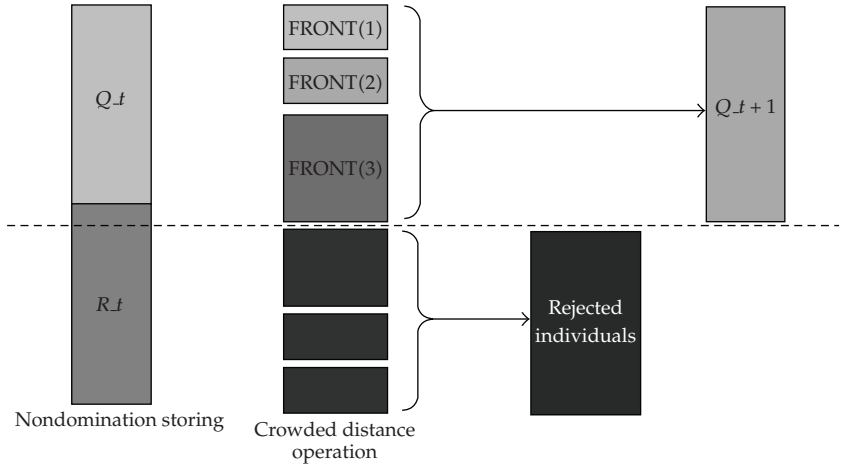

Figure 5: An outline of the NSGA-II procedure.

where $\beta_k (\geq 0)$ is a sample from a random number generated having the density

$$f_c(\beta) = \begin{cases} \dfrac{1}{2}(\eta_c + 1)\beta^{\eta_c} & 0 \leq \beta \leq 1, \\ \dfrac{1}{2}(\eta_c + 1)\dfrac{1}{\beta^{\eta_{c+2}}} & \beta \geq 1. \end{cases} \tag{6.10}$$

This distribution can be obtained from a $u \sim \mathcal{U}(0,1)$ source. η_c is the distribution index for the recombination operator.

6.2.9. Main Loop

First, a random parent population is created. The population is then sorted based on the nondomination outlined above. Each individual is assigned a fitness value (or rank) equal to its nondomination level, with 1 representing the best level. Binary tournament selection, recombination, and mutation operators are then applied to create an offspring population of size N_{pop}.

Due to the introduction of elitism, the current population is compared to previously found best nondominated solutions, the procedure is different after the initial generation. Figure 5 outlines the tth generation of the NSGA-II procedure.

The first step is to combined parent population Q_t with the offspring population R_t. The population is now of size $2N_{\text{pop}}$. The resulting population is then sorted according to nondomination. Since all previous and current population members are included in $N\_t = Q\_t \cup R\_t$, elitism is ensured.

Solutions belonging to the best nondominated set FRONT(1), are of best solutions in the combined population and must be emphasised more than any other solution in the combined population. If FRONT(1) is smaller than N_{pop}, then all individuals in this set are chosen for the next generation, else a truncation operation will have to be applied. If the first

case holds, then the remaining members of the next generation are chosen from subsequent nondominated fronts in the order of their ranking.

In general, the count of solutions in all sets from FRONT(1) to FRONT(l) would be larger than the population size. To choose exactly N_{pop} population members, the solutions of the last frontier are sorted and then the crowded-comparison operator is applied in descending order. The best solutions are chosen to fill all population slots. This procedure is valid for minimisation problems only.

The requisite diversity among nondominated solutions is introduced by using the crowded comparison procedure, which is used in the tournament selection and during the population truncation phases [4].

6.2.10. Initialisation and Termination Conditions

Initialisation is kept simple in most EA applications, the first population is seeded by randomly generated individuals. In principle, problem-specific heuristics can be used in this step, to create an initial population with higher fitness values [40].

We can distinguish two cases for an appropriate termination condition. If the problem has a known optimum fitness level, then an acceptable error bound would be a suitable condition. However, in many multiobjective problems optimum values are not known. In this case one needs to extend this condition with one that certainly stops the algorithm. As noted by Eiben and Smith [40], a common choice is to impose a limit on the total number of fitness evaluations.

6.3. Optimisation Problem

The optimal bespoke structure can be derived by solving the following problem:

$$\begin{aligned} &\min_{\mathbf{x}\in[0,1]^N} \Theta(\mathbf{x}), \\ &\max_{\mathbf{x}\in[0,1]^N} s_{n_t r}(\mathbf{x}) \\ \text{subject to} \quad & \\ &\sum_{i=1}^{N} x_i = 1, \end{aligned} \tag{6.11}$$

where in the above problem, we minimise the portfolio ETL, and simultaneously maximise the $n_{\text{tr}}^{\text{th}}$ tranche spread denoted by $s_{n_{\text{tr}}}$ subject to the portfolio weights summing up to unity. This minimality may typically be the case if only long positions in the underlying credits are allowed.

For practical purposes, it may be crucial to include other real-world constraints. Let $\mathcal{G}^*$ denote the set of organisational, regulatory and physical constraints. The list of constraints, which are commonly included into $\mathcal{G}^*$ are summarised below.

(1) *Upper and lower limits on issuer weights*: especially due to regulatory issues, the portfolio fractional amount invested into each asset may be limited, that is, there

may be individual lower and upper bounds on the portfolio weights.

$$l_{\text{upper}}^{(i)} \leq x_i \leq l_{\text{upper}}^{(i)} \quad \forall i \in \{1, 2, 3, \ldots, N\}. \tag{6.12}$$

When short selling is disallowed, it can be explicitly modelled by setting $l_i = 0$ in this formulation.

(2) *Regional/industry concentration constraints*: due to rating agency and regulatory restrictions, the portfolio may have industry and regional concentration limits, then

$$\sum_{i=1}^{k} x_i \leq U^{(I_{\text{ind}}, I_{\text{geo}})}, \quad \text{where } 0 \leq U^{(I_{\text{ind}}, I_{\text{geo}})} \leq 1, \tag{6.13}$$

where the coordinate $(I_{\text{ind}}, I_{\text{geo}})$ refers to a particular industry I_{ind} in region I_{geo}.

(3) *Cardinality constraints*: these constraints limit the number of assets in the portfolio. They can either be strict

$$\mathcal{N}(x_i > 0) = K^*, \quad 0 < K^* \leq a,\ K^* \in \mathbb{N}. \tag{6.14}$$

or lower and upper bounds on the cardinality can be defined,

$$K_l^* \leq \mathcal{N}(x_i > 0) \leq K_u^*, \quad 0 < K_l^* < K_u^* \leq a,\ K_l^*, K_u^* \in \mathbb{N}. \tag{6.15}$$

(4) *Weighted average rating factor (WARF) constraints*: due to regulatory or organisational restrictions, the portfolio WARF must be above some minimum value,

$$R_{\text{prot}} \geq R_{\text{target}}, \tag{6.16}$$

where in the case of S and P ratings $R_{\text{prot}} \in \{AAA, AA+, AA, AA-, \ldots\}$. (Due to the nature of CDOs, ratings higher than the portfolio rating can be achieved. In this case the constraint may be defined as $R_{\text{nth tranche}} \geq R_{\text{target}}$.)

Rating agencies presuppose that there are two broad credit considerations that determine the risk of a CDO portfolio: collateral diversity by industry and by reference entity, and the credit quality of each asset in the portfolio. With respect to the latter, rating agencies use the concept of the WARF. Each security in the CDO portfolio has a rating (actual or implied) and each rating has a corresponding rating factor. The lower a security's rating, the higher its corresponding rating factor. In order to calculate the weighted average debt rating for a pool of assets, take the par amount of each performing asset and multiply it by its respective rating factor. Then, sum the resulting amounts for all assets in the pool and divide this number by the sum of the par values of the performing assets [1].

Krahnen and Wilde [43] show that the common presumption about macro factor tail risk (extreme systematic risk) being largely held by senior tranches is false. While senior

tranches are in fact primarily exposed to tail risk, the reverse is not true, as the share of tail risk borne by senior tranches is quite limited. They conclude that tail risk in standard CDO transactions is held by all tranches. This argument motivates the use of a tail-risk measure like ETL in the tranche return-risk optimisation to derive optimal structures. The multiobjective optimisation problem including the above constraints is now defined in detail.

$$\begin{aligned}
&\min_{x\in\mathcal{X}} \mathrm{ETL}_{\beta_*}\left(L(t_j)\right),\\
&\max_{x\in\mathcal{X}} s_n = \frac{\sum_{j=1}^{J} D(0,(t_j - t_{j-1})/2)\left(\mathbb{E}[L_{n_{\mathrm{tr}}}(t_j)] - \mathbb{E}[L_{n_{\mathrm{tr}}}(t_{j-1})]\right)}{\sum_{j=1}^{J}(t_j - t_{j-1})D(0,t_j)\mathbb{E}[N_{\mathrm{out}}^{(n_{\mathrm{tr}})}(t_j,t_{j-1})]}\\
&\text{subject to}\\
&l_{\mathrm{upper}}^{(i)} \le x_i \le l_{\mathrm{upper}}^{(i)} \quad \forall i \in \{1,2,3,\ldots,N\},\\
&R_{\mathrm{prot}} \ge R_{\min},\\
&\mathcal{N}(x_i > 0) = 106.
\end{aligned} \tag{6.17}$$

We do not include the industry/region concentration constraints since the numerical analysis is conducted on the benchmark portfolio consisting of 106 constituents of the iTraxx Europe IG S5 index. The regional constraint will fall away, however, the industry constraint will still hold. The study will analyse the industry distributions of the resulting collateral portfolio's, for optimal [6,9]% mezzanine tranches that define the pareto frontier. This is an improtant visualisation tool for bespoke CDO investors, which depicts the trade-off between tranche return and risk, to allow for picking a structure given the respective risk preference. A flat Libor curve of 4.6% is applied. The CDO structure is priced as at 18-April-2008.

The two cases investigated are as follows.

(1) First an examination of the pareto frontiers for bespoke CDO structures under both Gaussian and Clayton copula assumptions is conducted. Long-only positions in the underlying CDS are allowed. The upper trading limit in any particular reference entity is set to 2.5% of the portfolio, whilst the lower limit is set to 0.5% to satisfy the cardinality constraint of having 106 credits in the collateral portfolio.

(2) Then an investigation into the behaviour of the pareto frontiers under increasing upper trading limit is conducted. This consequently allows an investigation of concentration risk and its effects on the producing optimal CDO structures. The upper trading limit is increased to 5% in this case.

Table 1 reprots the NSGA-II parameter setting used to solve the optimisation problem (6.17).

The parameters settings were found to be most appropriate after several runs of the algorithm. The computational time played a crucial role in the assignment of parameter values.

6.4. Prototype Experiment: Pareto Frontier Analysis

6.4.1. Optimal Structures with Long Only Credit portfolio

Bespoke CDOs are commonly preferred among investors because they can be used to execute a variety of customised investment objectives and strategies. Flexibility in choosing the

Table 1: NSGA-II parameter settings.

Evolutionary parameter	Value
Population size	100
Number of generations	500
Crossover probability	90%
Distribution index for crossover	0.01
Distribution index for mutation	0.01
Pool size	50
Tour size	2

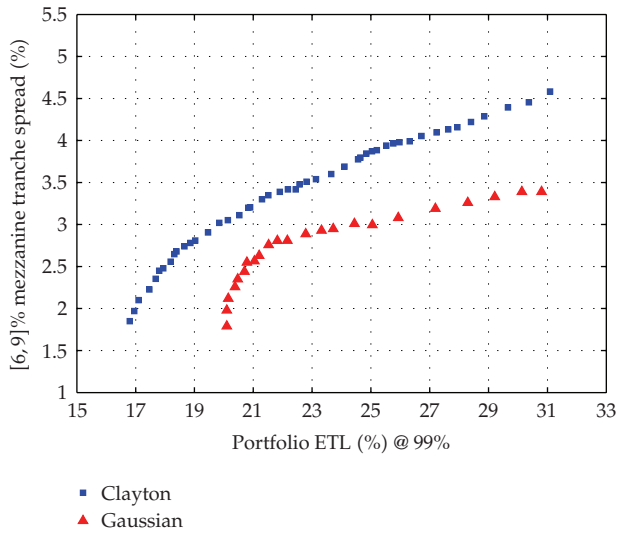

Figure 6: The comparison of bespoke CDO pareto frontiers under different default dependence assumptions.

reference portfolio allows investors to structure the portfolio investments such that it can be properly matched with their investment portfolios, as a separate investment strategy or as a hedging position. Most often investors choose credit portfolios such that diversification is only as a result of buying protection on reference entities in the underlying portfolio.

We derive the pareto frontiers solving problem (6.17) under both the Gaussian and Clayton copula assumptions when investors are restricted to the long-only condition.

Figure 6 displays the difference in pareto frontiers for the two default dependence assumptions. The first observation is that the Gaussian allocation is sub-optimal. For all levels of portfolio risk, measured by ETL, the Clayton allocation will result in higher $[6, 9]\%$ tranche spreads. This deviation is an increasing function of portfolio tail risk and widens to a maximum of 120 bps for an ETL of 30% of the portfolio losses.

The minimum tail risk portfolio can provide a $[6, 9]\%$ mezzanine tranche that pays 180 bps under the Clayton allocation, whilst the Gaussian allocation provides spread of 198 bps. However, there exist a 3.3% difference in ETL for these minimum risk portfolios. A similar spread under the Clayton allocation can be achieved with a ETL of just 16.95%.

We concluded that the asset allocation strategies based on the Gaussian copula will results in sub-optimal bespoke CDO structures.

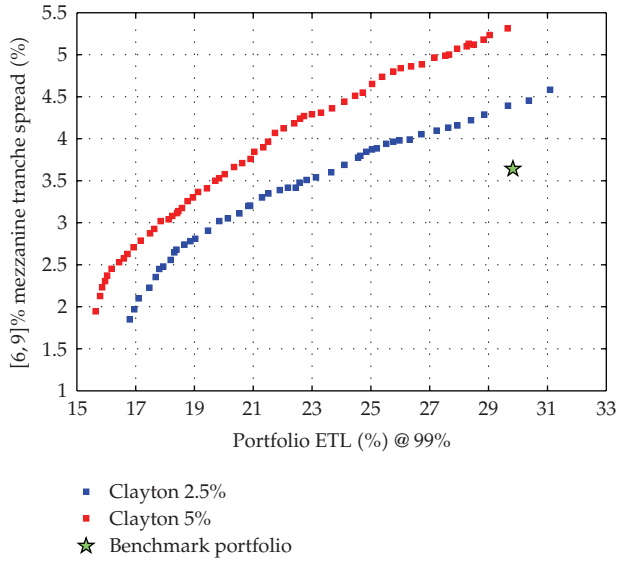

Figure 7: The portfolio notional distribution for the concentrated portfolio case, by issuer credit rating.

6.4.2. Issuer Concentrated Credit Structures

We now investigate the effects of issuer concentration on the pareto frontier. As we concluded in the previous chapter, that investors demand a higher premium for taking on concentration risk, combining this with the leverage effects of tranche technology, will result in higher tranche spreads than the well diversified case. The resulting frontiers are shown in Figure 7 for the $[6, 9]\%$ mezzanine tranche.

Figure 7 compares the resulting pareto-frontiers for the well-diversified and issuer concentrated cases. The CDO spread of the concentrated case over the well-diversified case is an increasing function of portfolio risk. For a 29.64% level of risk, a spread of 92 bps over the well-diversified case can be achieved.

7. Conclusions

The main objective of this research has been to develop and illustrate a methodology to derive optimal bespoke CDO structures using the NSGA-II technology. Until recently, the derivation of such structures was a near to an impossible task. However, with the advent of advance pricing and optimisation techniques applied in this research, credit structurers can use these tools to provide investors with optimal structures. Investors can use this type of methodology to compare similar rated credit structures and make an informed investment decision, especially under the current credit market conditions.

The most improtant finding is that the Gaussian copula allocation produces suboptimal CDO tranche investments. Better tranche returns at all levels of risk are obtained under the Clayton assumption. In the constrained long-only case, for all levels of portfolio risk, measured by ETL, the Clayton allocation will result in higher $[6, 9]\%$ tranche spread. This deviation was shown to be an increasing function of portfolio tail risk and widen to a maximum of 120 bps for an ETL of 30%.

It was also demonstrated that significant improvement of returns over standardised CDOs tranches of similar rating can be achieved using the methodology presented in this paper. The findings also concluded that the leverage effects should became more pronounced for tranches lower down the capital structure. More concentrated bespoke CDO portfolios result in higher tranche spreads. Investor demand higher premiums for taking on the concentration risk. For a 29.64% level of portfolio tail risk, an excess return of 92 bps over the well diversified case can be achieved.

Since the current analysis is restricted to single-factor copula models, one obvious extension for future studies should be on the added benefits multifactor copula models may provide in producing optimal CDO structures. On the use of evolutionary technology, a future study should compare NSGA-II results to the rival, the second generation of the strength pareto evolutionary algorithm (SPEA-2) proposed by the Zitzler et al. [44]. Stability of and convergence to the pareto frontier for optimal CDO structures should also be investigated.

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