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Research Article

Components of Pearson's Statistic for at Least Partially Ordered *m*-Way Contingency Tables

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For at least partially ordered three-way tables, it is well known how to arithmetically decompose Pearson's X_P^2 statistic into informative components that enable a close scrutiny of the data. Similarly well-known are smooth models for two-way tables from which score tests for homogeneity and independence can be derived. From these models, both the components of Pearson's X_P^2 and information about their distributions can be derived. Two advantages of specifying models are first that the score tests have weak optimality properties and second that identifying the appropriate model from within a class of possible models gives insights about the data. Here, smooth models for higher-order tables are given explicitly, as are the partitions of Pearson's X_P^2 into components. The asymptotic distributions of statistics related to the components are also addressed.

1. Introduction

In [1, 2] it is shown how, for at least partially ordered three-way tables, to arithmetically decompose Pearson's X_P^2 statistic into informative components that enable a close scrutiny of the data. They focus on three-way tables as being indicative of higher-order tables. Here, we give models for arbitrary multiway tables that are at least partially ordered. We discuss the arithmetic decomposition of X_P^2 into components, giving explicit formulae for these components. This enables X_P^2 to be partitioned into meaningful X^2-type statistics. Using extensions of models for two-way tables discussed in [3], the asymptotic distribution of statistics related to these components may be given.

At the onset, we should say what we mean by "ordered." A random variable is a mapping from the sample space to the real line. It is ordered if and only if the ordering of the range is meaningful. So, for example, a range containing only zero and one, denoting male and female, would not usually be considered meaningful. However, it would usually be considered meaningful if the zero and one denoted low and high, respectively. A variable is ordered if and only if it reflects a random variable that is ordered rather than not ordered, or nominal. A table is completely ordered if and only if all variables are ordered. It is partially ordered if and only if at least one but not all variables are ordered.

To give precedence, we observe here that the arithmetic decomposition of Pearson's X_P^2 statistic for two- and three-way tables can be shown quite compactly using results from [3, Chapter 4, Theorems 2.1 and 2.2, pages 90-91 and Theorem 5.2, page 101]. It is shown there that for contingency tables Lancaster's ϕ^2 is equal to the sum of the squares of the elements of a vector θ in the subsequent models. In a parallel manner, working with observed proportions $\{N_{ij}/n\}$, it can be shown that X_P^2 is equal to the sum of the squares of components. This observation applies to verifying the results in [1, 2]. Moreover, precedence should be given to the work in [3, Chapter 12] for material throughout this paper.

We also note that the work in [4] considered models for ordered two-way contingency tables. In [4, Chapter 3], an extended hypergeometric model is used when both row and column marginal totals are known. This is not a smooth model and here we will not discuss either it or its extensions further. In [4, Chapter 8], doubly ordered models are considered, and these will be generalised in Section 2 in what follows.

In treating a singly ordered table, the work in [4, Chapter 4] assumed the total count for each treatment is known before sighting the data, and this leads to a smooth product multinomial model. If the treatment totals are not known before sighting the data, the resulting model is a single multinomial with cell probabilities modelled the same way as when the treatment totals are known before sighting the data. The models in [1, 2] are single multinomials, following the second approach. However, it is clear that, in general, for partially ordered tables, there are a multitude of possible models, depending on which marginal totals are assumed known before sighting the data. In all cases, the logarithms of the likelihoods are, apart from unimportant constants, the same. Henceforth, we will consistently work with product multinomials and note that the distributional results developed apply to the multitude of models indicated.

The outline of this paper is as follows. In Section 2, the more routine case of completely ordered multiway tables is discussed. The balance of the paper is about the more complicated partially ordered tables. In Section 3, the work of [4] on partially ordered tables is reviewed. In Section 4, the work in [1, 2] is reviewed and extended using smooth models. Section 5 gives the generalizations to arbitrary multiway partially ordered tables.

2. Completely Ordered Multiway Tables

For an m-way $I_1 \times I_2 \times \cdots \times I_m$, completely ordered table of counts $\{N_{v_1 \cdots v_m}\}$, Pearson's X_P^2 is given by

$$X_P^2 = \sum_{v_1=1}^{I_1} \cdots \sum_{v_m=1}^{I_m} \frac{(N_{v_1 \cdots v_m} - E[N_{v_1 \cdots v_m}])^2}{E[N_{v_1 \ldots v_m}]}. \tag{2.1}$$

An extension of the approach in [1] demonstrates that X_P^2 has an arithmetic decomposition:

$$X_P^2 = \sum_{u_1=0}^{I_1-1} \cdots \sum_{u_m=0}^{I_m-1} Z_{u_1 \cdots u_m}^2, \tag{2.2}$$

in which the components $Z_{u_1\cdots u_m}$, $u_1 = 0,\ldots,I_1-1,\ldots, u_m = 0,\ldots\ ,I_m-1$, are given by

$$Z_{u_1\cdots u_m} = \sqrt{n}\sum_{v_1=1}^{I_1}\cdots\sum_{v_m=1}^{I_m} a_{u_1}(v_1)\cdots a_{u_m}(v_m)p_{v_1\cdots v_m}. \tag{2.3}$$

Here $n = \sum_{v_1=1}^{I_1}\cdots\sum_{v_m=1}^{I_m} N_{v_1\cdots v_m}$ and for $j = 1,\ldots,m$, $\{a_{u_j}(\bullet)\}$ is orthonormal on $\{p_{\bullet\cdots\bullet\ v_j\bullet\cdots\bullet}\}$, in which $p_{v_1\cdots v_m} = N_{v_1\cdots v_m}/n$ and $p_{\bullet\cdots\bullet v_j\bullet\cdots\bullet}$ is obtained from $p_{v_1\cdots v_m}$ by summing out all variables other than v_j. Furthermore, the orthonormal systems all have zeroth term identically one. This work builds on the iconic work of Oliver Lancaster, for which see [3], and [3, Chapter 12] in particular.

It is routine to show that the components $Z_{u_1\cdots u_m}$are asymptotically multivariate normal, since an arbitrary linear combination of these variables is asymptotically normal by the central limit theorem. Utilizing the orthonormality of the $\{a_{u_j}(\bullet)\}$, it can be shown that all components have expectation zero, variance unity, and covariances zero. They are thus asymptotically mutually independent and asymptotically standard normal.

One possible smooth model for $\{N_{v_1\cdots v_m}\}$ is the multinomial with count total n and cell probabilities $\{p_{v_1\cdots v_m}\}$ given by

$$p_{v_1\cdots v_m} = \left\{\sum_{u_1=0}^{I_1-1}\cdots\sum_{u_m=0}^{I_m-1}\theta_{u_1\cdots u_m}a_{u_1}(v_1)\cdots a_{u_m}(v_m)\right\}p_{v_1\bullet\cdots\bullet}\cdots p_{\bullet\cdots\bullet v_m}, \tag{2.4}$$

in which $\theta_{0\cdots 0} = 1$ and $\theta_{0\cdots 0u_j0\cdots 0} = 0$ for all $u_j \geq 1$. This model includes all genuine two, three, and so forth m-way independence models. A routine extension of [4, Theorem 8.1] shows that the score test statistic for testing, that the $\theta_{u_1\cdots u_m}$are collectively zero against the negation of this is, as before, the sum of the squares of the $Z_{u_1\cdots u_m}$. Moreover, these components have the distributional properties given in the previous paragraph. Generalising [4, Theorem 8.2], this score test statistic is X_P^2. The score test has the advantage of weak optimality: see, for example, [5]. An additional advantage of this approach is that it can be shown that $Z_{u_1\cdots u_m}^2$ is the score test statistic when testing $\theta_{u_1\cdots u_m} = 0$ against $\theta_{u_1\cdots u_m} \neq 0$ in an appropriate model. Thus, in an informal sense, every $Z_{u_1\cdots u_m}$ is a detector of the corresponding $\theta_{u_1\cdots u_m}$.

The degrees of freedom associated with X_P^2 are the number of $\theta_{u_1\cdots u_m}$ (and hence $Z_{u_1\cdots u_m}$) in the model, excluding those that are by convention always zero or one. The degrees of freedom are thus

$$\begin{aligned}&\prod_{i<j}(I_i-1)(I_j-1)+\prod_{i<j<k}(I_i-1)(I_j-1)(I_k-1)\cdots+(I_1-1)(I_2-1)\cdots(I_m-1)\\ &\quad= I_1\times I_2\times\cdots\times I_m-1-(I_1-1)-(I_2-1)-\cdots-(I_m-1).\end{aligned} \tag{2.5}$$

The left-hand side consists of the degrees of freedom associated with all genuine two-way, three-way, and so forth m-way models, while the right-hand side is the number of cells minus one for the constraint $n = \sum_{v_1=1}^{I_1}\cdots\sum_{v_m=1}^{I_m} N_{v_1\cdots v_m}$ (reflecting that the sample size is known before sighting the data) minus the degrees of freedom associated with all one-way (essentially goodness of fit) models. For the happiness example in [1] $I_1 = 3$, $I_2 = 4$, $I_3 = 5$ and substituting in the aforementioned formulae, there are 50 degrees of freedom.

3. Two-Way Singly Ordered Tables

In [4, Section 4.4] two-way tables are discussed. We report on that discussion using our subsequent convention that ordered categories precede unordered categories. Tables $\{N_{wz}\}$ are modelled by product multinomials, with the zth column being multinomial with total counts $n_{\bullet z}$ and cell probabilities:

$$p_{wz} = \left\{1 + \sum_{u=1}^{I_1} \theta_{uz} a_u(w) / \sqrt{n_{\bullet z}}\right\} p_{w\bullet}, \tag{3.1}$$

for $w = 1, \ldots, I_{1-1}$. Note that the probabilities in the I_1th row are found by difference: $p_{I_1 z} = 1 - p_{1z} - \cdots - p_{(I_1-1)z}$ and $z = 1, \ldots, I_2$, where $p_{w\bullet} = \sum_z N_{wz}/n$ in which $n = \sum_w \sum_z N_{wz}$. The efficient score contains random variables $Z_{uz} = \sqrt{(n/n_{\bullet z})} \sum_{w=1}^{I_1} a_u(w) p_{wz}$ and the information matrix is found to be singular. In order to find a score test statistic in [4, Section 4.4], the model is modified by removing the θs corresponding to the last column because the model is overparameterised: in any row, given the probabilities in the first $I_2 - 1$ columns and the marginal probability for that row (the average of all probabilities in that row), the probability corresponding to the final column can be readily determined. A quicker approach is now outlined.

Write $Z_u = (Z_{u1}, \ldots, Z_{uI_2})^T$ and $Z^T = (Z_1^T, \ldots, Z_{I_1-1}^T)$. The $n \times n$ identity matrix is written as I_n; this will be clear from the context when this, and not the number of rows, and so forth, is intended. From the information matrix for Z, the covariance matrix for Z_u is $I_{I_2} - (\sqrt{n_{\bullet a} n_{\bullet b}})/n$. This is idempotent of rank $I_2 - 1$. There exists an $I_2 \times (I_2 - 1)$ matrix A such that $I_{I_2} - (\sqrt{n_{\bullet a} n_{\bullet b}})/n = AA^T$ and $A^T A = I_{I_2 - 1}$. We now focus on a smooth model containing just one value of u (the full model is similar). Since the information matrix in terms of $\theta_u = (\theta_{u1}, \ldots, \theta_{uI_2})^T = \theta$ say is singular, define ϕ by $A\phi = \theta$. Then using the results of the lemma in [6, Section 3], the efficient score and information in terms of θ (U_θ and I_θ) and ϕ (U_ϕ and I_ϕ) are related by $U_\phi = A^T U_\theta$ and $I_\phi = A^T I_\theta A$, respectively. It follows that since, in terms of ϕ, the efficient score is $A^T Z_u = Y_u$ say, and the information matrix is $A^T\{I_{I_2} - (\sqrt{n_{\bullet a} n_{\bullet b}})/n\}A = I_{I_2-1}$, the score test statistic in terms of ϕ is $Y_u^T Y_u = Z_u^T\{I_{I_2} - (\sqrt{n_{\bullet a} n_{\bullet b}})/n\}Z_u$. Since Y_u is asymptotically $\mathrm{N}_{I_2-1}(0, I_{I_2-1})$, the score test statistic has the $\chi^2_{I_2-1}$ distribution, as is otherwise well known.

The columns of A are eigenvectors corresponding to the nonzero eigenvalues of $\{I_{I_2} - (\sqrt{n_{\bullet a} n_{\bullet b}})/n\}$. The eigenvector corresponding to the zero eigenvalue is $(1, \ldots, 1)^T$, so a typical eigenvector may be written $1_\perp$. The elements of $Y_u = A^T Z_u$ are of the form $1_\perp^T Z_u$, that may fairly be called a contrast between the elements of Z_u. They are mutually independent and standard normal. While the Z_{ui} are immediately interpretable, they are slightly less convenient than Y_{ui} that are orthogonal contrasts and are asymptotically mutually independent and asymptotically standard normal. These contrasts correspond to each order u, $u = 1, \ldots, I_1 - 1$, and reflect comparisons between the levels of the unordered factor. They may, for example, compare the means of the first two levels, the mean of the first two levels with that of the third level, the mean of the first three levels with that of the fourth level, and so on. Such contrasts may be described as Helmertian, from the Helmert matrix. In its simplest form, the Helmert matrix is an orthogonal $(n+1) \times (n+1)$ matrix with all the elements of the first row $1/\sqrt{(n+1)}$ and rth row $1/\sqrt{[r(r+1)]}$ (r times), $-r/\sqrt{[r(r+1)]}$, then all zeros.

4. Three-Way Partially Ordered Tables

4.1. Singly Ordered Three-Way Tables

For singly ordered $I_1 \times I_2 \times I_3$ tables, a product multinomial model is assumed, with the counts corresponding to the z_1th column and z_2th layer, $z_1 = 1, \ldots, I_2$ and $z_2 = 1, \ldots, I_3$, being multinomial with total counts $n_{\bullet z_1 z_2}$ and cell probabilities:

$$p_{wz_1z_2} = p_{w\bullet\bullet} \sum_{u=0}^{I_1-1} \theta_{uz_1z_2} a_u(w), \tag{4.1}$$

for $w = 1, \ldots, I_1$, in which $\theta_{0z_1z_2} = 1$. Here and henceforth, the normalisation corresponding to the $\sqrt{n_{\bullet z}}$ factor in p_{wz} in Section 3 is absorbed into the $\theta_{uz_1z_2}$. The components are random variables:

$$Z_{uz_1z_2} = \sqrt{\frac{n}{p_{\bullet z_1 \bullet} p_{\bullet\bullet z_2}}} \sum_{w=1}^{I_1} a_u(w) p_{wz_1z_2}, \tag{4.2}$$

where $p_{\bullet z_1 \bullet} = \sum_w \sum_{z_2} N_{wz_1z_2}/n$ and $p_{\bullet\bullet z_2} = \sum_w \sum_{z_1} N_{wz_1z_2}/n$. The $Z_{uz_1z_2}$ are immediately interpretable [2]), and, by the multivariate central limit theorem, are asymptotically multivariate normal. This does not depend on the smooth model. As in Section 3, for each u, $u = 1, \ldots, I_1 - 1$, we may construct orthogonal contrasts that are asymptotically mutually independent and asymptotically standard normal. These contrasts reflect uth moment comparisons between the levels of the unordered factors.

In [2], without a model, it is shown that X_P^2 is the sum of the squares of the $Z_{uz_1z_2}$:

$$X_P^2 = \sum_{u=0}^{I_1-1} \sum_{z_1=1}^{I_2} \sum_{z_2=1}^{I_3} Z_{uz_1z_2}^2. \tag{4.3}$$

In X_P^2, it is insightful to separate components corresponding to $u = 0$ and $u \neq 0$. Thus

$$X_P^2 = \sum_{z_1=1}^{I_2} \sum_{z_2=1}^{I_3} Z_{0z_1z_2}^2 + \sum_{u=1}^{I_1-1} \sum_{z_1=1}^{I_2} \sum_{z_2=1}^{I_3} Z_{uz_1z_2}^2. \tag{4.4}$$

The first summand corresponds to a two-way completely unordered table obtained by summing over rows and may reasonably be denoted by $X_{Z_1Z_2}^2$. The second summation corresponds to a genuinely three-way singly ordered table and may reasonably be denoted by $X_{UZ_1Z_2}^2$.

In [2] it is stated that the degrees of freedom associated with X_P^2 are $I_1I_2I_3 - I_1 - I_2 - I_3 + 2$. This follows because there are $(I_2 - 1)(I_3 - 1)$ degrees of freedom associated with $X_{Z_1Z_2}^2$, and $(I_1 - 1)(I_2I_3 - 1)$ degrees of freedom associated with $X_{UZ_1Z_2}^2$.

We can argue for these degrees of freedom by, when possible, counting the $\theta_{uz_1z_2}$ or the $Z_{uz_1z_2}$. The table corresponding to $X_{Z_1Z_2}^2$ is completely unordered, so there are no $\theta_{uz_1z_2}$ to count. We propose no smooth model, and our components are not appropriate when there is no order. However, the degrees of freedom are known independently to be $(I_2 - 1)(I_3 - 1)$.

The table corresponding to $X^2_{UZ_1Z_2}$ has degrees of freedom $(I_1 - 1)(I_2I_3 - 1)$ since this is the number of parameters $\theta_{uz_1z_2}$ in the smooth model. There are I_2I_3 multinomials, each of which has $(I_1 - 1)$ parameters $\theta_{uz_1z_2}$ as the multinomials probabilities sum to one (so the final cell probability is given by difference). In addition, one of the I_2I_3 multinomials is determined by $\{p_{w\bullet\bullet}\}$ and the remaining multinomials.

4.2. Doubly Ordered Three-Way Tables

For doubly ordered tables a product multinomial model is again assumed, with the counts corresponding to the zth layer being multinomial with total counts $n_{\bullet\bullet z}$ and cell probabilities:

$$p_{w_1w_2z} = p_{w_1w_2\bullet} \sum_{u_1=0}^{I_1-1} \sum_{u_2=0}^{I_2-1} \theta_{u_1u_2z} a_{u_1}(w_1) a_{u_2}(w_2), \tag{4.5}$$

for $w_1 = 1, \ldots, I_1$, $w_2 = 1, \ldots, I_2$, and $z = 1, \ldots, I_3$, in which $\theta_{00z} = \theta_{u_10z} = \theta_{0u_2z} = 1$. The components are random variables:

$$Z_{u_1u_2z} = \sqrt{\frac{n}{p_{\bullet\bullet z}}} \sum_{w_1=1}^{I_1} \sum_{w_2=1}^{I_2} a_{u_1}(w_1) a_{u_2}(w_2) p_{w_1w_2z} \tag{4.6}$$

for $u_1 = 0, \ldots, I_1 - 1$, $u_2 = 0, \ldots, I_2 - 1$, and $z = 1, \ldots, I_3$, where $p_{\bullet\bullet z} = \sum_{w_1} \sum_{w_2} N_{w_1w_2z}/n$. Again, by the multivariate central limit theorem, the $Z_{u_1u_2z}$ are asymptotically multivariate normal. This does not depend on the smooth model. For each (u_1, u_2) pair, as in Section 3, we may construct orthogonal contrasts that are asymptotically mutually independent and asymptotically standard normal. These contrasts reflect bivariate moment comparisons between the levels of the unordered factor. A typical contrast may be (1st, 2nd) moment differences between the first two levels reflected by layers.

In [2], without a model, it is shown that X^2_P is the sum of the squares of the $Z_{u_1u_2z}$:

$$X^2_P = \sum_{u_1=0}^{I_1-1} \sum_{u_2=0}^{I_2-1} \sum_{z=1}^{I_3} Z^2_{u_1u_2z}. \tag{4.7}$$

Again in X^2_P, it is insightful to separate components corresponding to $u_i = 0$ and $u_i \neq 0$. Thus,

$$X^2_P = \sum_{z=1}^{I_3} Z^2_{00z} + \sum_{u_1=1}^{I_1-1} \sum_{z=1}^{I_3} Z^2_{u_10z} + \sum_{u_2=1}^{I_2-1} \sum_{z=1}^{I_3} Z^2_{0u_2z} + \sum_{u_1=1}^{I_1-1} \sum_{u_2=1}^{I_2-1} \sum_{z=1}^{I_3} Z^2_{u_1u_2z}. \tag{4.8}$$

The first summand is identically zero. The second summand corresponds to a two-way singly ordered table obtained by summing over columns and may reasonably be denoted by $X^2_{U_1Z}$. The third summation corresponds to another two-way singly ordered table obtained by summing over rows and may reasonably be denoted by $X^2_{U_2Z}$. The final summation corresponds to a genuine three-way doubly ordered table and may reasonably be denoted by $X^2_{U_1U_2Z_2}$.

In [2] it is incorrectly claimed that the associated degrees of freedom are, as in Section 4.1, $I_1I_2I_3 - I_1 - I_2 - I_3 + 2$. The one-way table corresponding to the components with $u_1 = u_2 = 0$ is uninformative, and should be ignored. The two-way tables corresponding to precisely one of the u_1 or u_2 zero are single-ordered, and, as in Section 3, have degrees of freedom $(I_1 - 1)(I_3 - 1)$ and $(I_2 - 1)(I_3 - 1)$, respectively. When neither u_1 nor u_2 is zero, the corresponding table is a genuine doubly ordered three-way table. There are I_3 multinomials, each with $(I_1 - 1)(I_2 - 1)$ parameters $\theta_{u_1u_2z}$ in their smooth model, but in fact the final of the I_3 multinomials is determined by the $\{p_{\bullet\bullet z}\}$ and the remaining multinomials. So there are $(I_1 - 1)(I_2 - 1)(I_3 - 1)$ degrees of freedom for this final table. In all, the degrees of freedom are

$$(I_1 - 1)(I_2 - 1)(I_3 - 1) + (I_1 - 1)(I_2 - 1) + (I_1 - 1)(I_3 - 1) = (I_1I_2 - 1)(I_3 - 1). \tag{4.9}$$

We note that although the degrees of freedom in the Happiness Example of [2] are in error, the P values and conclusions with the correct degrees of freedom are as given there. We recommend the reader refer to this example, examined from two different perspectives in [1, 2], to see the insight and interpretability the components give to data analysis.

5. m-Way Partially Ordered Tables

We consider now an m-way table that is at least partially ordered: without loss of generality the first r (≥ 1) categorical variables are taken as ordered and the remaining $s = m - r$ (≥ 1) categorical variables are nominal. The notation reflects this convention; the subscripts w reflect ordered categories while the subscripts z reflect nominal categories. Accordingly, the table is denoted by $\{N_{w_1\cdots w_r z_1\cdots z_s}\}$. As in [2] and [4, Chapter 4], we define components of the form

$$Z_{u_1\cdots u_r z_1\cdots z_s} = \sqrt{\frac{n}{\{p_{\bullet\cdots\bullet z_1\bullet\cdots\bullet} \times \cdots \times p_{\bullet\cdots\bullet z_s}\}}} \sum_{w_1=1}^{I_1} \cdots \sum_{w_r=1}^{I_r} a_{u_1}(w_1) \cdots a_{u_r}(w_r) p_{w_1\cdots w_r z_1\cdots z_s}, \tag{5.1}$$

where $p_{w_1\cdots w_r z_1\cdots z_s} = N_{w_1\cdots w_r z_1\cdots z_s}/n$ and where $\{a_{u_j}(\bullet)\}$, $p_{\bullet\cdots\bullet z_j\bullet\cdots\bullet}$ and $p_{w_1\cdots w_r z_1\cdots z_s}$ are defined similarly to the above. Again, by the multivariate central limit theorem, the $Z_{u_1\cdots u_r z_1\cdots z_s}$ are asymptotically multivariate normal. This does not depend on the smooth model.

By manipulations similar to those for the three-way case, it is possible to argue that

$$X^2_P = \sum_{u_1=0}^{I_1-1} \cdots \sum_{u_r=0}^{I_r-1} \sum_{z_1=1}^{I_{r+1}} \cdots \sum_{z_s=1}^{I_m} Z^2_{u_1\cdots u_r z_1\cdots z_s}. \tag{5.2}$$

By separating components corresponding to $u_i = 0$ and $u_i \neq 0$, X_P^2 can be partitioned as follows:

$$X_P^2 = \sum_{z_1=1}^{I_{r+1}} \cdots \sum_{z_s=1}^{I_m} Z^2_{0\cdots 0 z_1 \cdots z_s} + \sum_{i=1}^{r} \sum_{u_i=1}^{I_i-1} \sum_{z_1=1}^{I_{r+1}} \cdots \sum_{z_s=1}^{I_m} Z^2_{0\cdots 0 u_i 0 \cdots 0 z_1 \cdots z_s}$$
$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{r} \sum_{u_i=1}^{I_i-1} \sum_{u_j=1}^{I_j-1} \sum_{z_1=1}^{I_{r+1}} \cdots \sum_{z_s=1}^{I_m} Z^2_{0\cdots 0 u_i 0 \cdots 0 u_j 0 \cdots 0 z_1 \cdots z_s} + \cdots \tag{5.3}$$
$$+ \sum_{u_1=1}^{I_1-1} \cdots \sum_{u_r=1}^{I_r-1} \sum_{z_1=1}^{I_{r+1}} \cdots \sum_{z_s=1}^{I_m} Z^2_{u_1 \cdots u_r z_1 \cdots z_s}.$$

If $s = 1$, the first term corresponds to a noninformative one-way table and contributes zero to the sum. The following term corresponds to all $(s+1)$-way singly ordered tables obtained by summing over $r-1$ ordered marginals and may reasonably be denoted by $\sum_{i=1}^{r} X^2_{U_i Z_1 \cdots Z_s}$. The following term corresponds to all $(s+2)$-way doubly ordered tables obtained by summing over $r-2$ ordered marginals and may reasonably be denoted by

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{r} X^2_{U_i U_j Z_1 \cdots Z_s}. \tag{5.4}$$

The subsequent terms involve components with successively more ordered marginals and correspond to tables that are of increasing size. The final term corresponds to a genuine m-way r-fold ordered table and may reasonably be denoted by $X^2_{U_1 \cdots U_r Z_1 \cdots Z_s}$. Thus,

$$X_P^2 = \sum_{i=1}^{r} X^2_{U_i Z_1 \cdots Z_s} + \sum_{\substack{i,j=1 \\ i \neq j}}^{r} X^2_{U_i U_j Z_1 \cdots Z_s} + \cdots + X^2_{U_1 \cdots U_r Z_1 \cdots Z_s}. \tag{5.5}$$

The smooth model envisaged here is product multinomial where for each $(z_1, \ldots, z_s)$, the observations follow a multinomial distribution with total counts $n_{\bullet \cdots \bullet z_1 \cdots z_s}$ and cell probabilities $\{p_{w_1 \cdots w_r z_1 \cdots z_s}\}$ given by

$$p_{w_1 \cdots w_r z_1 \cdots z_s} = p_{w_1 \bullet \cdots \bullet} \times \cdots \times p_{\bullet \cdots \bullet w_r \bullet \cdots \bullet} \left\{ \sum_{u_1=0}^{I_1-1} \cdots \sum_{u_r=0}^{I_r-1} \theta_{u_1 \cdots u_r z_1 \cdots z_s} a_{u_1}(w_1) \cdots a_{u_r}(w_r) \right\}. \tag{5.6}$$

An extension of the approach in [4, Section 4.4] investigates testing if the $\theta_{u_1 \cdots u_r z_1 \cdots z_s}$ are collectively zero against the negation of this. Generalising the work in [4, Section 4.4], the efficient score statistic is $Z_{w_1 \cdots w_r z_1 \cdots z_s}$. The information matrix is block diagonal but each block is singular. Nevertheless, the efficient score is asymptotically normal and appropriate orthogonal contrasts are asymptotically mutually independent and standard normal.

The degrees of freedom may be deduced either by counting $\theta_{u_1 \cdots u_r z_1 \cdots z_s}$ (or the corresponding components), or by the arguments in [2]. Consider a genuine m-way table with the first r categories ordered and the remaining $s = m - r$ categories not ordered. This includes tables corresponding to $X^2_{U_i U_j Z_1 \cdots Z_s}$ say, resulting from summing out several of the ordered variables. This is now a doubly ordered $(s+2)$-way table. The degrees of freedom for $X^2_{U_1 \cdots U_r Z_1 \cdots Z_s}$ are $(I_1 - 1)(I_2 - 1) \cdots (I_r - 1)(I_{r+1} \times I_{r+2} \times \cdots \times I_m - 1)$. There are $I_{r+1} \times I_{r+2} \times \cdots \times I_m$ multinomials (corresponding to $Z_1 = z_1, \ldots, Z_s = z_s$) each with $(I_1 - 1)(I_2 - 1) \cdots (I_r - 1)$ degrees of freedom. However, one of these multinomials is determined by the marginals and the other multinomial models.

We decline to write out the contrasts corresponding to the asymptotically mutually independent standard normal variables that are linear combinations of the $Z_{w_1 \cdots w_r z_1 \cdots z_s}$. The approach is similar to that employed in Section 3.

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