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Research Article

An Inventory System for Deteriorating Products with Ramp-Type Demand Rate under Two-Level Trade Credit Financing

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An inventory system for deteriorating products, with ramp-type demand rate, under two-level trade credit policy is considered. Shortages are allowed and partially backlogged. Sufficient conditions of the existence and uniqueness of the optimal replenishment policy are provided, and an algorithm, for its determination, is proposed. Numerical examples highlight the obtained results, and sensitivity analysis of the optimal solution with respect to major parameters of the system is carried out.

1. Introduction

In the conventional economic order quantity (EOQ) model, it is assumed that the supplier is paid for the items immediately after the items are received. In practice, the supplier may provide to the retailer a permissible delay in payments. During this credit period, the retailer can accumulate the revenue and earn interest on that revenue. However, beyond this period the supplier charges interest on the unpaid balance. Hence, a permissible delay indirectly reduces the cost of holding stock. On the other hand, trade credit offered by the supplier encourages the retailer to buy more. Thus it is also a powerful promotional tool that attracts new customers, who consider it as an alternative incentive policy to quantity discounts. Hence, trade credit can play a major role in inventory control for both the supplier as well as the retailer (see Jaggi et al. [1]). Three types of trade credit have been appeared, mainly, in the literature:

(i) a fixed trade credit period, (Goyal [2] Aggarwal and Jaggi [3], Jamal et al. [4], Chang and Dye [5], Teng [6], Jaber [7], Jaggi et al. [1], Ouyang and Cheng [8], Chung and Huang [9]);

(ii) a two-level trade credit known as $r/M_1/M_2$. More precisely, the supplier provides r discount off the price if the payment is made within period M_1; otherwise, the full payment is due within period M_2. (Huang [10], Liao [11], Teng and Chang [12]);

(iii) a trade credit period linked to the ordering quantity (Chang et al. [13], Chung and Liao [14], Ouyang et al. [15]).

For a comprehensive review for inventory lot-size models under trade credits, the reader is referred to Chang et al. [16].

In the literature referring to models with permissible delay in payments, the demand is, mostly, treated either as constant or as continuous differentiable function of time. However, in the case of a new brand of consumer good coming to the market, its demand rate increases in its growth stage (i.e., $[0, \mu]$) and then remains stable in its maturity stage (i.e., $[\mu, T]$). In addition, the demand rate of a seasonable product increases at the beginning of the season up to a certain moment (say, μ) and then remains constant for the rest of the planning horizon, T. The term "ramp-type" is used to represent such demand pattern. Hill [17] proposed an inventory model with variable branch being any power function of time. Research on this field continues with Mandal and Pal [18], Wu and Ouyang [19], and Wu [20]. In the above-cited papers, the optimal replenishment policy requires to determine the decision time (say, t_1) at which the inventory level falls to zero. Consequently, the following two cases should be examined: (1) the inventory level fall to zero before the demand reaches constant (i.e., $t_1 < \mu$) and (2) the inventory level falls to zero after the demand reaches constant (i.e., $t_1 > \mu$). Almost all of the researchers examined only the first case. Deng et al. [21] first reconsidered the inventory models proposed by Mandal and Pal [18] and Wu and Ouyang [19] and discussed both cases. Panda et al. [22] developed an inventory model for deteriorating items (with three-parameter Weibull distributed deterioration rate) with generalized exponential ramp-type demand rate and complete backlogging. Skouri et al. [23] extend the work of Deng et al. [21] by introducing a general ramp-type demand rate and Weibull deterioration rate. Panda et al. [24] presented a production-inventory model with generalized quadratic ramp-type demand rate and constant deterioration rate when shortages are not allowed. Skouri and Konstantaras [25] extended their previous work [23] studying an order level inventory model for deteriorating items based on time-dependent three branches ramp-type demand rate. Lin [26] studied an inventory model with general ramp-type demand rate, constant deterioration rate, complete backlogging, and several replenishment cycles during the finite time and used the hide-and-seek simulated annealing (SA) approach to determine the optimal replenishment policy.

This paper is an extension of the inventory system of Skouri et al. [23] assuming constant deterioration rate, when the two-level trade credit scheme, $r/M_1/M_2$, which was described above, is considered. The study of this system requires the examination of the ordering relations between the time parameters M_1, M_2, μ, T, which, actually, lead to the following different models:

(i) $M_1 \leq \mu < M_2 < T$, (ii) $M_1 < M_2 \leq \mu < T$, (iii) $\mu \leq M_1 < M_2 < T$, (iv) $\mu \leq M_1 < T < M_2$, (v) $\mu < T \leq M_1 < M_2$, (vi) $M_1 \leq \mu < T \leq M_2$.

Note that from the definition of demand rate $\mu < T$ and from credit scheme $M_1 < M_2$.

This study can be used: (1) for the determination of the optimal replenishment policy under a specific trade credit settings (corresponding to one of the six models mentioned above) and (2) for supplier' selection, since it is obvious that the ordering of the parameters μ, M_1, M_2, T leads to different trade credit offers. Although the analysis of all models is available upon request, in order to reduce the length of the paper, only the first model will be presented.

The paper is organized as follows: the notation and assumptions used are given in Section 2. In Section 3, the quantities and functions, which are common to each of the possible models are derived. The mathematical formulation of the first model and the determination of the optimal policy are provided in Section 4. In Section 5, numerical examples highlighting the results obtained are given, and sensitivity analysis with respect to major parameters of the system is carried out. The paper closes with concluding remarks in Section 6.

2. Notation and Assumptions

The following notation is used through the paper.

2.1. Notation

T is the constant scheduling period (cycle),

t_1 the time when the inventory level falls to zero,

S the maximum inventory level at each scheduling period (cycle),

C_p the unit purchase cost,

c_1 the inventory holding cost per unit per unit time,

c_2 the shortage cost per unit per unit time,

c_3 the cost incurred from the deterioration of one unit,

c_4 the per unit opportunity cost due to the lost sales ($c_4 > C_p$ see Teng et al. [27]),

p the unit selling price,

I_e the interest rate earned,

I_c the interest rate charged,

r cash discount rate, $0 < r < 1$,

M_1 the period of cash discount in years,

M_2 the period of permissible delay in payments in years, $M_1 < M_2$,

μ the parameter of the ramp-type demand function (time point), and

$I(t)$ the inventory level at time t.

2.2. Assumptions

The inventory model is developed under the following assumptions.

(1) The ordering quantity brings the inventory level up to the order level S. Replenishment rate is infinite.

(2) Shortages are backlogged at a rate $\beta(x)$ which is a nonincreasing function of x with $0 < \beta(x) \leq 1$, $\beta(0) = 1$ and x is the waiting time up to the next replenishment. Moreover, it is assumed that $\beta(x)$ satisfies the relation $C_2\beta(x) + C_2T\beta'(x) + C_p\beta'(x) \geq 0$, where $\beta'(x)$ is the derivate of $\beta(x)$. The case with $\beta(x) = 1$ corresponds to complete backlogging model.

(3) The supplier offers cash discount if payment is paid within M_1; otherwise, the full payment is paid within M_2, (see Huang [10]).

(4) The on-hand inventory deteriorates at a constant rate θ $(0 < \theta < 1)$ per time unit. The deteriorated items are withdrawn immediately from the warehouse and there is no provision for repair or replacement.

(5) The demand rate $D(t)$ is a ramp-type function of time given by

$$D(t) = \begin{cases} f(t), & t < \mu, \\ f(\mu), & t \geq \mu, \end{cases} \tag{2.1}$$

where $f(t)$ is a positive, differentiable function of $t \in (0, T]$.

3. Deriving the Common Quantities for the Inventory Models

In this section, common quantities entering to all models will be derived. Note that these quantities are affected only by the ordering relations between t_1 and μ. The inventory level $I(t)$, $0 \leq t \leq T$ satisfies the following differential equations:

$$\frac{dI(t)}{dt} + \theta I(t) = -D(t), \quad 0 \leq t \leq t_1 \tag{3.1}$$

with boundary condition $I(t_1) = 0$ and

$$\frac{dI(t)}{dt} = -D(t)\beta(T-t), \quad t_1 \leq t \leq T \tag{3.2}$$

with boundary condition $I(t_1) = 0$.

From the two possible relations between parameters t_1 and μ, (i) $t_1 \leq \mu$ and (ii) $t_1 > \mu$, and following identical steps as in Skouri et al. [23], the sum of holding, deterioration, shortages, and lost sales cost is obtained as

$$C(t_1) = \begin{cases} C_1(t_1) & \text{if } t_1 \leq \mu, \\ C_2(t_1) & \text{if } t_1 > \mu, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} C_1(t_1) = c_1 &\left\{ \int_0^{t_1} e^{-\theta t} \left[\int_t^{t_1} f(x) e^{\theta x} dx \right] dt \right\} \\ &+ c_2 \left\{ \int_{t_1}^{\mu} (\mu - t) f(t) \beta(T-t) dt + f(\mu) \int_{\mu}^{T} \left[\int_{\mu}^{t} \beta(T-x) dx \right] dt \right. \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mu}^{T}\left[\int_{t_1}^{\mu} f(x)\beta(T-x)dx\right] dt\Bigg\} \\
&+c_3\left\{\int_0^{t_1} f(t)e^{\theta t}dt-\int_0^{t_1} f(t)dt\right\} \\
&+c_4\left\{\int_{t_1}^{\mu}(1-\beta(T-t))f(t)dt+f(\mu)\int_{\mu}^{T}(1-\beta(T-t))dt\right\},
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
C_2(t_1)=c_1&\left\{\int_0^{\mu} e^{-\theta t}\left[\int_t^{\mu} f(x)e^{\theta x}dx+f(\mu)\int_{\mu}^{t_1} e^{\theta x}dx\right]dt\right. \\
&\left.+f(\mu)\int_{\mu}^{t_1} e^{-\theta t}\left[\int_t^{t_1} e^{\theta x}dx\right]dt\right\} \\
&+c_3\left\{\int_0^{\mu} f(t)e^{\theta t}dt+f(\mu)\int_{\mu}^{t_1} e^{\theta t}dt-\int_0^{\mu} f(t)dt-f(\mu)(t_1-\mu)\right\} \\
&+c_2\left\{f(\mu)\int_{t_1}^{T}(T-x)\beta(T-x)dx\right\}+c_4\left\{f(\mu)\int_{t_1}^{T}(1-\beta(T-t))dt\right\}.
\end{aligned}
\tag{3.5}
$$

4. Model I—The Inventory Model When $M_1 \le \mu < M_2 < T$

In order to obtain the total cost for this model, the purchasing cost, interest charges for the items kept in stock, and the interest earned should be taken into account.

Since the supplier offers cash discount if payment is paid within M_1, there are two payment policies for the buyer. Either the payment is paid at time M_1 to receive the cash discount (Case 1) or the payment is paid at time M_2 so as not to receive the cash discount (Case 2). Then, these two cases will be discussed.

Case 1 (payment is made at time M_1). In this case, the following subcases should be considered.

Subcase 1.1 ($t_1 \le M_1 \le \mu < T$). The purchasing cost is

$$
C_{A1,1}(t_1)=C_p(1-r)\left[\int_0^{t_1} f(x)e^{\theta x}dx+f(\mu)\int_{\mu}^{T}\beta(T-x)dx+\int_{t_1}^{\mu} f(x)\beta(T-x)dx\right]. \tag{4.1}
$$

The interest earned during the period of positive inventory level is

$$
I_{T1,1}(t_1)=pI_e\int_0^{t_1}\int_0^{t} f(x)dx\,dt+pI_e(M_1-t_1)\int_0^{t_1} f(x)dx. \tag{4.2}
$$

Since $t_1 \leq \mu$, the total cost in the time interval $[0, T]$ is calculated using (3.4), (4.1), and (4.2)

$$TC_{11}(t_1) = C_1(t_1) + C_{A1,1}(t_1) - I_{T1,1}(t_1). \tag{4.3}$$

Subcase 1.2 ($M_1 < t_1 \leq \mu < T$). The purchasing cost is $C_{A1,1}$ (relation (4.1)).

The interest payable for the inventory not being sold after the due date M_1 is

$$P_{T2,1}(t_1) = C_p(1-r)I_c \int_{M_1}^{t_1} e^{-\theta t}\left[\int_t^{t_1} f(x)e^{\theta x}\right] dx\, dt. \tag{4.4}$$

The interest earned is

$$I_{T2,1}(t_1) = pI_e \int_0^{t_1}\int_0^t f(x)dxdt. \tag{4.5}$$

Since again $t_1 \leq \mu$, the total cost over $[0, T]$ is calculated using the relations (3.4), (4.1), (4.4), and (4.6) and is

$$TC_{12}(t_1) = C_1(t_1) + C_{A1,1}(t_1) + P_{T2,1}(t_1) - I_{T2,1}(t_1). \tag{4.6}$$

Subcase 1.3 ($M_1 \leq \mu \leq t_1 \leq T$). The purchasing cost is

$$C_{A2,1}(t_1) = C_p(1-r)\left[\int_0^{\mu} f(x)e^{\theta x}dx + f(\mu)\int_{\mu}^{t_1} e^{\theta x}dx + f(\mu)\int_{t_1}^{T} \beta(T-x)dx\right]. \tag{4.7}$$

The interest earned, $I_{T3,1}$, is:

$$I_{T3,1}(t_1) = pI_e\left(\int_0^{\mu}\int_0^t f(x)dx\, dt + \int_{\mu}^{t_1}\int_0^{\mu} f(x)dx\, dt + \int_{\mu}^{t_1}\int_{\mu}^{t} f(\mu)dx\, dt\right). \tag{4.8}$$

The interest payable for the inventory not being sold after the due date M_1 is

$$\begin{aligned} P_{T3,1}(t_1) = C_p(1-r)I_c\Bigg(&\int_{M_1}^{\mu} e^{-\theta t}\int_t^{\mu} e^{\theta x} f(x)dx\, dt \\ &+ f(\mu)\int_{M_1}^{\mu} e^{-\theta t}\int_{\mu}^{t_1} e^{\theta x}dx\, dt + f(\mu)\int_{\mu}^{t_1} e^{-\theta t}\int_t^{t_1} e^{\theta x}dx\, dt\Bigg). \end{aligned} \tag{4.9}$$

Since $\mu < t_1$, the total cost over $[0, T]$ is again calculated from (3.5), (4.7)–(4.9) and is

$$TC_{13}(t_1) = C_2(t_1) + C_{A2,1}(t_1) + P_{T3,1}(t_1) - I_{T3,1}(t_1). \tag{4.10}$$

The results obtained lead to the following total cost function:

$$TC_1(t_1) = \begin{cases} TC_{1,1}(t_1), & t_1 \le M_1, \\ TC_{1,2}(t_1), & M_1 < t_1 \le \mu, \\ TC_{1,3}(t_1), & \mu \le t_1. \end{cases} \tag{4.11}$$

So the problem is

$$\min_{t_1} TC_1(t_1). \tag{4.12}$$

Its solution requires, separately, studying each of three branches and then combining the results to obtain the optimal policy. It is easy to check that $TC_1(t_1)$ is continuous at the points M_1 and μ.

The first-order condition for a minimum of $TC_{1,1}(t_1)$ is

$$\begin{aligned} \frac{dTC_{1,1}(t_1)}{dt_1} &= \left\{ \frac{c_1 + c_3\theta}{\theta}\left(e^{\theta t_1} - 1\right) - c_2(T - t_1)\beta(T - t_1) \right. \\ &\quad \left. - c_4\left(1 - \beta(T - t_1)\right) - pI_e(M_1 - t_1) + C_p(1 - r)\left(e^{\theta t_1} - \beta(T - t_1)\right) \right\} f(t_1) = 0. \end{aligned} \tag{4.13}$$

Since $dTC_{1,1}(0)/dt_1 < 0$ and $dTC_{1,1}(T)/dt_1 > 0$, (4.13) has at least one root. So if $t_{1,1}$ is the root of (4.13), this corresponds to minimum since

$$\begin{aligned} \left.\frac{dTC_{1,1}^2(t_1)}{dt_1^2}\right|_{t_1 = t_{1,1}} &= f(t_{1,1})\left\{ (c_1 + c_3\theta)e^{\theta t_{1,1}} + c_2\beta(T - t_{1,1}) + c_2(T - t_{1,1})\beta'(T - t_{1,1}) \right. \\ &\quad \left. - c_4\beta'(T - t_{1,1}) + pI_e + C_p(1 - r)\left(\theta e^{\theta t_{1,1}} + \beta'(T - t_{1,1})\right) \right\} > 0. \end{aligned} \tag{4.14}$$

Consequently, $t_{1,1}$ is the unique unconstrained minimum of $TC_{1,1}(t_1)$.

The first-order condition for a minimum of $TC_{1,2}(t_1)$ is

$$\begin{aligned} \frac{dTC_{1,2}(t_1)}{dt_1} &= \frac{c_1 + c_3\theta}{\theta}\left(e^{\theta t_1} - 1\right) f(t_1) - c_2(T - t_1)\beta(T - t_1) f(t_1) - c_4\left(1 - \beta(T - t_1)\right) f(t_1) \\ &\quad + C_p(1 - r)\left(e^{\theta t_1} - \beta(T - t_1)\right) + \frac{C_p(1 - r)I_c}{\theta}\left(e^{\theta(t_1 - M_1)} - 1\right) f(t_1) \\ &\quad - pI_e\int_0^{t_1} f(x)dx = 0. \end{aligned} \tag{4.15}$$

Let us set $h(x) = c_2 x\beta(x) + c_4(1-\beta(x)) - pI_e x$. If $t_{1,2}$ is the root of (4.15) (this may or may not exist), $f(x)$ is an increasing function and further if $h'(x) > 0$, then

$$\begin{aligned}\left.\frac{d^2TC_{1,2}(t_1)}{dt_1^2}\right|_{t_1=t_{1,2}} &= pI_e f'(t_{1,2})\int_0^{t_{1,2}} f(x)dx \\ &\quad + f(t_{1,2})\Big\{(c_1+c_3\theta)e^{\theta t_{1,2}} \;\; + c_2\left[\beta(T-t_{1,2}) + (T-t_{1,2})\beta'(T-t_{1,2})\right] \\ &\quad - c_4\beta'(T-t_{1,2}) + C_p(1-r)I_c e^{\theta(t_{1,2}-M_1)} \\ &\quad + C_p(1-r)\left(\theta e^{\theta t_{1,2}} + \beta'(T-t_{1,2})\right) - pI_e\Big\} > 0,\end{aligned} \tag{4.16}$$

and this $t_{1,2}$ corresponds to unconstrained minimum of $TC_{1,2}(t_1)$.

The first-order condition for a minimum of $TC_{1,3}(t_1)$ is

$$\begin{aligned}\frac{dTC_{1,3}(t_1)}{dt_1} &= f(\mu)\Big\{\frac{c_1+c_3\theta}{\theta}\left(e^{\theta t_1}-1\right) - c_2(T-t_1)\beta(T-t_1) - c_4(1-\beta(T-t_1)) \\ &\quad - pI_e(t_1-\mu) + \frac{C_p(1-r)I_c}{\theta}\left(e^{\theta(t_1-M_1)}-1\right) + C_p(1-r)\left(e^{\theta t_1} - \beta(T-t_1)\right)\Big\} \\ &\quad - pI_e\int_0^{\mu} f(x)dx = 0.\end{aligned} \tag{4.17}$$

If $t_{1,3}$ is the root of (4.17) (this may or may not exist) and $h'(x) > 0$, then

$$\begin{aligned}\frac{d^2TC_{1,3}(t_1)}{dt_1^2} &= \Big((c_1+c_3\theta)e^{\theta t_1} + c_2\left[\beta(T-t_1) + (T-t_1)\beta'(T-t_1)\right] - c_4\beta'(T-t_1) \\ &\quad - pI_e + C_p(1-r)I_c e^{\theta(t_1-M_1)} + C_p(1-r)\left(\theta e^{\theta t_1} + \beta'(T-t_1)\right)\Big) f(\mu) > 0,\end{aligned} \tag{4.18}$$

this $t_{1,3}$ corresponds to unconstrained minimum of $TC_{1,3}(t_1)$.

Remark 4.1. The function $TC_1(t_1)$ is not differentiable in M_1.

Then, the following procedure summarizes the previous results for the determination of the optimal replenishment policy, when payment is made at time M_1.

Step 1. Find the global minimum of $TC_{1,1}(t_1)$, say $t^*_{1,1,M_1}$, as follows.

Substep 1.1. Compute $t_{1,1,M_1}$ from (4.13); if $t_{1,1,M_1} < M_1$, then set $t^*_{1,1,M_1} = t_{1,1,M_1}$ and compute $TC_{1,1}(t^*_{1,1,M_1})$ else go to Substep 1.2.

Substep 1.2. Find the $\min\{TC_{1,1}(0), TC_{1,1}(M_1)\}$ and accordingly set $t^*_{1,1,M_1}$.

Step 2. Find the global minimum of $TC_{1,2}(t_1)$, say $t^*_{1,2,M_1}$, as follows.

Substep 2.1. Compute $t_{1,2,M_1}$ from (4.15); if $M_1 < t_{1,2,M_1} < \mu$, then set $t^*_{1,2,M_1} = t_{1,2,M_1}$ and compute $TC_2(t^*_{1,2,M_1})$ else go to Substep 2.2.

Substep 2.2. Find the $\min\{TC_{1,2}(M_1), TC_{1,2}(\mu)\}$ and accordingly set $t^*_{1,2,M_1}$.

Step 3. Find the global minimum of $TC_{1,3}(t_1)$, say $t^*_{1,3,M_1}$, as follows.

Substep 3.1. Compute $t_{1,3,M_1}$ from (4.17); if $\mu < t_{1,3,M_1}$, then set $t^*_{1,3,M_1} = t_{1,3,M_1}$and compute $TC_{1,3}(t^*_{1,3,M_1})$ else go to Substep 3.2.

Substep 3.2. Find the $\min\{TC_{1,3}(\mu), TC_{1,3}(T)\}$ and accordingly set $t^*_{1,3,M_1}$.

Step 4. Find the $\min\{TC_{1,1}(t^*_{1,1,M_1}), TC_{1,2}(t^*_{1,2,M_1}), TC_{1,3}(t^*_{1,3,M_1})\}$ and accordingly select the optimal value for t_1 say t_{1,M_1} with optimal cost $C_1(t_{1,M_1})$.

Case 2 (payment is made at time M_2). When the payment is made at time M_2 the following cases should be considered.

Subcase 2.1 ($t_1 \leq \mu < M_2 < T$). The purchasing cost is.

$$C_{A1,2}(t_1) = \frac{C_{A1,1}(t_1)}{1-r}. \tag{4.19}$$

The interest earned during the period of positive inventory level is.

$$I_{T1,2}(t_1) = pI_e \int_0^{t_1} \int_0^{t} f(x)dx\,dt + pI_e(M_2 - t_1) \int_0^{t_1} f(x)dx. \tag{4.20}$$

Since $t_1 \leq \mu$, the total cost in the time interval $[0,T]$ is calculated using (3.4), (4.19), and (4.20)

$$TC_{2,1}(t_1) = C_1(t_1) + C_{A1,2}(t_1) - I_{T1,2}(t_1). \tag{4.21}$$

Subcase 2.2 ($\mu < t_1 \leq M_2 < T$). The purchasing cost is

$$C_{A2,2} = \frac{C_{A2,1}(t_1)}{1-r}. \tag{4.22}$$

The interest earned is

$$\begin{aligned} I_{T2,2}(t_1) = pI_e\bigg[&\int_0^{\mu} \int_0^{t} f(x)dx\,dt + \int_{\mu}^{M_2} \int_0^{\mu} f(x)dx\,dt + \int_{\mu}^{t_1} \int_{\mu}^{t} f(\mu)dx\,dt \\ &+ \int_{t_1}^{M_2} \int_{\mu}^{t_1} f(\mu)dx\,dt\bigg]. \end{aligned} \tag{4.23}$$

Since again $\mu \leq t_1$, the total cost over $[0, T]$ is calculated using the relations (3.5), (4.22), and (4.23) and is

$$TC_{2,2}(t_1) = C_2(t_1) + C_{A2,2}(t_1) - I_{T2,2}(t_1). \tag{4.24}$$

Subcase 2.3 ($\mu \leq M_2 \leq t_1 \leq T$). The purchasing cost is $C_{A2,2}(t_1)$.

The interest earned, $I_{T3,2}$, is

$$I_{T3,2}(t_1) = pI_e\left(\int_0^{\mu}\int_0^{t} f(x)dx\,dt + \int_{\mu}^{t_1}\int_0^{\mu} f(x)dx\,dt + \int_{\mu}^{t_1}\int_{\mu}^{t} f(\mu)dx\,dt\right). \tag{4.25}$$

The interest payable for the inventory not being sold after the due date M_2 is

$$P_{T3,2}(t_1) = C_pI_cf(\mu)\int_{M_2}^{t_1} e^{-\theta t}\int_t^{t_1} e^{\theta x}dx\,dt. \tag{4.26}$$

Since $\mu < t_1$, the total cost over $[0, T]$ is again calculated from (3.5), (4.22), (4.25), and (4.26) and is

$$TC_{2,3}(t_1) = C_2(t_1) + C_{A2,2}(t_1) + P_{T3,2}(t_1) - I_{T3,2}(t_1). \tag{4.27}$$

The results obtained lead to the following total cost function:

$$TC_2(t_1) = \begin{cases} TC_{2,1}(t_1), & t_1 \leq \mu, \\ TC_{2,2}(t_1), & \mu < t_1 \leq M_2, \\ TC_{2,3}(t_1), & M_2 \leq t_1. \end{cases} \tag{4.28}$$

So the problem is

$$\min_{t_1} TC_2(t_1). \tag{4.29}$$

Its solution, as in the previous case, requires, separately, studying each of three branches and then combining the results to obtain the optimal policy. It is easy to check that $TC_2(t_1)$ is continuous at the points M_2 and μ.

The first-order condition for the minimum for $TC_{2,1}(t_1)$ is

$$\begin{aligned}\frac{dTC_{2,1}(t_1)}{dt_1} = \Bigg\{ &\frac{c_1 + c_3\theta}{\theta}\left(e^{\theta t_1} - 1\right) - c_2(T - t_1)\beta(T - t_1) - c_4(1 - \beta(T - t_1)) \\ &-pI_e(M_2 - t_1) + C_p\left(e^{\theta t_1} - \beta(T - t_1)\right)\Bigg\} f(t_1) = 0.\end{aligned} \tag{4.30}$$

Since $dTC_{2,1}(0)/dt_1 < 0$ and $dTC_{2,1}(T)/dt_1 > 0$, (4.30) has at least one root. So if $t_{1,1}$ is the root of (4.30), this corresponds to minimum as

$$\left.\frac{dTC_{2,1}^2(t_1)}{dt_1^2}\right|_{t_1=t_{1,1}} = f(t_{1,1})\Big\{(c_1+c_3\theta)e^{\theta t_{1,1}} + c_2\beta(T-t_{1,1}) + c_2(T-t_{1,1})\beta'(T-t_{1,1}) \\ -c_4\beta'(T-t_{1,1}) + pI_e + C_p\Big(\theta e^{\theta t_{1,1}} + \beta'(T-t_{1,1})\Big)\Big\} > 0. \tag{4.31}$$

So $t_{1,1}$ is the unconstrained minimum of $TC_{2,1}(t_1)$.

The first-order condition for a minimum of $TC_{2,2}(t_1)$ is

$$\frac{dTC_{2,2}(t_1)}{dt_1} = f(\mu)\Big\{\frac{c_1+c_3\theta}{\theta}\Big(e^{\theta t_1}-1\Big) - c_2(T-t_1)\beta(T-t_1) - c_4(1-\beta(T-t_1)) \\ -pI_e(M_2-t_1) + C_p\Big(e^{\theta t_1} - \beta(T-t_1)\Big)\Big\} = 0. \tag{4.32}$$

If $t_{1,2}$ is the root of (4.32) (this may or may not exist), this corresponds to unconstrained minimum of $TC_{2,2}(t_1)$ as

$$\left.\frac{dTC_{2,2}^2(t_1)}{dt_1^2}\right|_{t_1=t_{1,2}} = f(\mu)\Big\{(c_1+c_3\theta)e^{\theta t_{1,2}} + c_2\beta(T-t_{1,2}) + c_2(T-t_{1,2})\beta'(T-t_{1,2}) \\ -c_4\beta'(T-t_{1,2}) + pI_e + C_p\Big(\theta e^{\theta t_{1,2}} + \beta'(T-t_{1,2})\Big)\Big\} > 0. \tag{4.33}$$

The first-order condition for a minimum of $TC_{2,3}(t_1)$ is

$$\frac{dTC_{2,3}(t_1)}{dt_1} = f(\mu)\Big\{\frac{c_1+c_3\theta}{\theta}\Big(e^{\theta t_1}-1\Big) - c_2(T-t_1)\beta(T-t_1) - c_4(1-\beta(T-t_1)) - pI_e(t_1-\mu) \\ +\frac{C_pI_c}{\theta}\Big(e^{\theta(t_1-M_2)}-1\Big) + C_p\Big(e^{\theta t_1} - \beta(T-t_1)\Big)\Big\} - pI_e\int_0^{\mu} f(x)dx = 0. \tag{4.34}$$

If $t_{1,3}$ is a root of (4.34) (this may or may not exist) and $c_1 + c_3\theta + C_pI_c \ge pI_e$ this corresponds to unconstrained minimum of $TC_{2,3}(t_1)$ as

$$\frac{dTC_{2,3}^2(t_1)}{dt_1^2} = \Big\{(c_1+c_3\theta+C_pI_c)e^{\theta t_1} + c_2\beta(T-t_1) + c_2(T-t_1)\beta'(T-t_1) \\ -c_4\beta'(T-t_1) - pI_e + C_p\Big(\theta e^{\theta t_1} + \beta'(T-t_1)\Big)\Big\}f(\mu). \tag{4.35}$$

Remark 4.2. The function $TC_2(t_1)$ is not differentiable in M_2.

The procedure for the determination of the optimal replenishment policy when payment is made at time M_2 is as follows.

Step 1. Find the global minimum of $TC_{2,1}(t_1)$, say $t^*_{1,1,M_2}$, as follows.

Substep 1.1. Compute $t_{1,1,M_2}$ from (4.30); if $t_{1,1,M_2} < \mu$, then set $t^*_{1,1,M_2} = t_{1,1,M_2}$ and compute $TC_{2,1}(t^*_{1,1,M_2})$ else go to Substep 1.2.

Substep 1.2. Find the $\min\{TC_{2,1}(0), TC_{2,1}(\mu)\}$ and accordingly set $t^*_{1,1,M_2}$.

Step 2. Find the global minimum of $TC_{2,2}(t_1)$, say $t^*_{1,2,M_2}$, as follows.

Substep 2.1. Compute $t_{1,2,M_2}$ from (4.32); if $\mu < t_{1,2,M_2} < M_2$, then set $t^*_{1,2,M_2} = t_{1,2,M_2}$ and compute $TC_{2,2}(t^*_{1,2,M_2})$ else go to Substep 2.2.

Substep 2.2. Find the $\min\{TC_{2,2}(\mu), TC_{2,2}(M_2)\}$ and accordingly set $t^*_{1,2,M_2}$.

Step 3. Find the global minimum of $TC_{2,3}(t_1)$, say $t^*_{1,3,M_2}$, as follows.

Substep 3.1. Compute $t_{1,3,M_2}$ from (4.34); if $M_2 < t_{1,3,M_2} < T$, then set $t^*_{1,3,M_2} = t_{1,3,M_2}$ and compute $TC_{2,3}(t^*_{1,3,M_2})$ else go to Substep 3.2.

Substep 3.2. Find the $\min\{TC_{2,3}(M_2), TC_{2,3}(T)\}$ and accordingly set $t^*_{1,3,M_2}$.

Step 4. Find the $\min\{TC_{2,1}(t^*_{1,1,M_2}), TC_{2,2}(t^*_{1,2,M_2}), TC_{2,3}(t^*_{1,3,M_2})\}$ and accordingly select the optimal value for t_1 say t_{1,M_2} with optimal cost $TC_2(t_{1,M_2})$.

Finally to find the overall optimum t_1 for the problem under consideration, the results obtained for the two presented cases (i.e., payment is made at M_1 and payment is made at M_2) are combined, that is, find $\min\{TC_1(t_{1,M_1}), TC_2(t_{1,M_2})\}$ and accordingly select the optimal value t^*_1.

5. Numerical Examples and Sensitivity Analysis

In this section, a numerical example is provided to illustrate the results obtained in previous sections. In addition, a sensitivity analysis, with respect to some important model's parameters, is carried out.

The input parameters are $c_1 = 3$€ per unit per unit time, $c_2 = 15$€ per unit per unit time, $c_3 = 5$€ per unit, $c_4 = 20$€ per unit per unit time, $r = 0.005$, $\mu = 0.3$ years, $\theta = 0.001$, $T = 0.5$ years, $f(t) = 3e^{4.5t}$ and $\beta(x) = e^{-0.2x}$, $M_1 = 0.13$ years, $M_2 = 0.43$ years, $p = 15$, $C_p = 10$, $I_e = 0.12$, $I_c = 0.15$.

5.1. The Payment Is Made at M_1

From (4.13), $t_{1,1,M_1} = 0.399$, which is not feasible as $t_{1,1,M_1} > M_1$. Since $TC_{1,1}(0) = 55.469$ and $TC_{1,1}(M_1) = 51.9633$, it follows that $t^*_{1,1,M_1} = M_1$. From (4.15), $t_{1,2,M_1} = 0.426$, which is not valid again as $t_{1,2,M_1} > \mu$. Since $TC_{1,1}(M_1) = TC_{1,2}(M_1) = 51.9633$ and $TC_{1,2}(\mu) = 46.6669$, the optimal value for $t^*_{1,2,M_1} = \mu$. From (4.17) $t_{1,3,M_1} = 0.423$; this value for t_1 is valid as $\mu < t_{1,3,M_1} < T$ so $t^*_{1,3,M_1} = t_{1,3,M_1}$ and $TC_{1,3}(t^*_{1,3,M_1}) = 44.8287$.

Finally $TC_{1,3}(t^*_{1,3,M_1}) = \min\{TC_{1,1}(M_1), TC_{1,2}(\mu), TC_{1,3}(t^*_{1,3,M_1})\} = 44.8287$ and consequently $t^*_{1,M_1} = 0.423$.

Table 1: Sensitivity analysis: the effect of changing the parameter (i) keeping all other parameters unchanged.

Parameter (i)	Percentage of changes (%)	t_1^*	*Time of payment*	$TC(t_1^*)$
μ	−50	0.46	M_2	27.0538
	−25	0.44	M_2	44.4206
	+25	0.45	M_2	52.8972
	+50	0.45	M_2	58.8673
M_1	−50	0.451	M_2	44.3039
	−25	0.451	M_2	44.3039
	+25	0.451	M_2	44.3039
	+50	0.451	M_2	44.3039
M_2	−50	0.434	M_2	44.4075
	−25	0.443	M_2	44.4206
	+25	0.43	M_2	43.6966
	+50	0.43	M_2	43.5449
r	−50	0.451	M_2	44.3039
	−25	0.451	M_2	44.3039
	+25	0.451	M_2	44.3039
	+50	0.451	M_2	44.3039

5.2. The Payment Is Made at M_2

From (4.30), $t_{1,1,M_2} = 0.424$ which is not feasible as $t_{1,1,M_2} > \mu$. Since $TC_{2,1}(0) = 55.6719$ $\kappa\alpha\iota$ $TC_{2,1}(\mu) = 46.2334$, it follows that $t^*_{1,1,M_2} = \mu$. From (4.32), $t_{1,2,M_2} = 0.424$ which is valid again as $\mu < t_{1,2} < M_2$ so $t^*_{1,2,M_2} = 0.424$ and $TC_{2,2}(t^*_{1,2,M_2}) = 44.3497$. From (4.34), $t_{1,3,M_2} = 0.451$; this value for t_1 is also valid as $\mu < M_2 < t_{1,3} < T$ so $t^*_{1,3,M_1} = t_{1,3,M_1}$ and $TC_{2,3}(t^*_{1,3,M_1}) = 44.3039$.

Finally $TC_{2,3}(t^*_{1,3,M_2}) = \min\{TC_{2,1}(\mu), TC_{2,2}(t^*_{1,3,M_2}), TC_{2,3}(t^*_{1,3,M_2})\} = 44.3039$, and consequently $t^*_{1,M_2} = 0.451$.

So, as $TC_2(t_{1,M_2}) = \min\{TC_1(t_{1,M_1}), TC_2(t_{1,M_2})\}$, the optimal t_1 is $t_1^* = t^*_{1,M_2} = 0.451$, which leads to a payment at M_2.

Using the data of the previous example, a sensitivity analysis is carried out to explore the effect of change on some, of the basic, model's parameters (μ, M_1, M_2, T, r) to the optimal policy (i.e., t_1 time of payment and optimal total cost). The results are presented in Table 1 and some interesting findings are summarized as follows.

(1) The changes of parameters M_1, M_2 and r have no impact on the optimal t_1, the time of payment and the optimal cost.

(2) The error on the parameters' estimation of μ has no impact on the time of payment, small impact on the optimal t_1, but high impact on the total optimal cost. This last observation is in line with the relative findings in Deng et al. [21].

6. Conclusions

In this paper, the following interrelated factors, which have appeared in the literature of inventory control, are incorporated: (i) the product's life cycle, which implies that its demand can be described as a ramp-type function of time, (ii) the effect of deterioration, (iii) the

$r/M_1/M_2$ credit scheme, which can be offered by supplier to the retailer for stimulating the demand, and (iv) the diminished, with the waiting time, backlogging rate, which is described as a decreasing function of time. As a result, this paper is a modification of the inventory system presented by Skouri et al. [23] when the $r/M_1/M_2$ credit scheme is considered. The study of this system requires the examination of the ordering relations between the time parameters M_1, M_2, μ, T, which, actually, lead to the six different models. This inventory system, setting $f(t) = D_0 t$, $\beta(x) = 1$, $M_1 = M_2 = 0$, $I_p = 0$, and $I_c = 0$, can give as special cases the ones presented by Mandal and Pal [18], Wu and Ouyang [19], and Deng et al. [21]. This model could be extended assuming several replenishment cycles during the planning horizon. For this extension, the application of some popular heuristic optimization algorithm (like Particle Swarm Optimization or Differential Evolution) may be useful, [28–30].

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