Hindawi Publishing Corporation
Advances in Decision Sciences
Volume 2012, Article ID 289035, 14 pages
doi:10.1155/2012/289035

*Research Article*

# An Allocation Scheme for Estimating the Reliability of a Parallel-Series System

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Received 28 April 2012; Revised 28 July 2012; Accepted 8 October 2012

Academic Editor: Wai Ki Ching



We give a hybrid two-stage design which can be useful to estimate the reliability of a parallel-series and/or by duality a series-parallel system. When the components' reliabilities are unknown, one can estimate them by sample means of Bernoulli observations. Let $T$ be the total number of observations allowed for the system. When $T$ is fixed, we show that the variance of the system reliability estimate can be lowered by allocation of the sample size $T$ at components' level. This leads to a discrete optimization problem which can be solved sequentially, assuming $T$ is large enough. First-order asymptotic optimality is proved systematically and validated *via* Monte Carlo simulation.

## 1. Introduction

In reliability engineering two crucial objectives are considered: (1) to maximize an estimate of system reliability and (2) to minimize the variance of the reliability estimate. Because system designers and users are risk averse, they generally prefer the second objective which leads to a system design with a slightly lower reliability estimate but a lower variance of that estimate, (e.g., [1]). It provides decision makers efficient rules compared to other designs which have a higher system reliability estimate, but with a high variability of that estimate. In the case of parallel-series and/or by duality series-parallel systems, the variance of the reliability estimate can be lowered by allocation of a fixed sample size (the number of observations or units tested in the system), while reliability estimate is obtained by testing components, see Berry [2]. Allocation schemes for estimation with cost, see, for example, [2–7], lead generally to a discrete optimization problem which can be solved sequentially using adaptive designs in a fixed or a Bayesian framework. Based on a decision theoretic approach, the authors seek to minimize either the variance or the Bayes risk associated to a squared error loss function. The problem of optimal reliability estimation reduces to a problem of optimal allocation of the sample sizes between Bernoulli populations. Such problems can be solved *via* dynamic programming but this technique becomes costly and intractable for complex systems. In the case of a two component series or parallel system, optimal procedures can be obtained and solved analytically when the coefficients of variation of the associated Bernoulli populations are known, (cf., e.g., [5, 8]). Unfortunately, these coefficients are not known in practice since they depend themselves on the unknown components' reliabilities of the system. In [9], the author has defined a sequential allocation scheme in the case of a series system and has shown its first-order asymptotic optimality for large sample sizes with comparison to the balanced scheme. In [10], a reliability sequential scheme (R-SS) was applied successfully to parallel-series systems, when the total number of units to be tested in each subsystem was fixed. Recently, in [11], a two-stage design for the same purpose was presented and shown to be asymptotically optimal when the subsystems sample sizes are fixed and large at the same order of the total sample size of the system. The problem considered in this paper is useful to estimate the reliability of a parallel-series and/or by duality a series-parallel system, when the components' reliabilities are unknown as well as the total numbers of units allowed to be tested in each subsystem. This work improves the results in [11] by developing a hybrid two-stage design to get a dynamic allocation between the sample sizes allowed for subsystems and those allowed for their components. For example, consider a parallel system of four components (1), (2), (3), and (4), with reliabilities 0.05, 0.1, 0.95, and 0.99, respectively, under the constraint that the total number of observations allowed is $T = 100$. Then, the sequential scheme given in [10] suggests to test, respectively, 10, 10, 28, and 52 units and produces a variance of the system reliability estimate equal $10^{-7}$, approximately. This is visibly better, compared to the balanced scheme which takes an allocation equal 25 in each component and produces a variance ten times greater then the former. The hybrid sequential scheme proposed in this paper is a tool to solve the same problem when the components are replaced by subsystems. More precisely, it combines the schemes developed for parallel and/or series systems in order to obtain approximately the best allocation at subsystems' level as well as at components' level.

In Section 2, definitions and preliminary results are presented accompanied by the proper two-stage design for a parallel subsystem just as was defined in [11] and its asymptotic optimality is proved for a fixed and large sample size. In Section 3, a parallel-series system is considered and it is shown that the variance of its reliability estimate has a lower bound independent of allocation. This leads, in Section 4, to the main result of this paper which lies in the hybrid two-stage algorithm and its asymptotic optimality for a fixed and large sample size allowed for the system. In Section 5, the results are validated *via* Monte Carlo simulation and it is shown that our algorithm leads asymptotically to the best allocation scheme to reach the lower bound of the variance of the reliability estimate. The last section is reserved for conclusion and remarks.

## 2. Preliminary Results

Consider a system $S$ of $n$ subsystems $S_1, S_2, \ldots, S_n$ connected in series, each subsystem $S_j$ contains $n_j$ components $S_{1j}, S_{2j}, \ldots, S_{n_j j}$ connected in parallel. The system should be referred to as parallel-series system. Assume $s$-independence within and across populations, then the system reliability is

$$R = \prod_{j=1}^{n} R_j, \tag{2.1}$$

where

$$R_j = 1 - \prod_{i=1}^{n_j} \left(1 - R_{ij}\right) \tag{2.2}$$

is the reliability of the parallel subsystem $S_j$ and $R_{ij}$ the reliability of the component $S_{ij}$. An estimator of $R$ is assumed to be the product of sample reliabilities:

$$\widehat{R} = \prod_{j=1}^{n} \widehat{R}_j, \tag{2.3}$$

where

$$\widehat{R}_j = 1 - \prod_{i=1}^{n_j} \left(1 - \widehat{R}_{ij}\right), \tag{2.4}$$

and $\widehat{R}_{ij}$ is the sample mean of functioning units in component $S_{ij}$:

$$\widehat{R}_{ij} = \frac{\sum_{l=1}^{M_{ij}} X_{ij}^{(l)}}{M_{ij}}. \tag{2.5}$$

$\widehat{R}_{ij}$ is used to estimate $R_{ij}$ where $M_{ij}$ is the sample size and $X_{ij}^{(l)}$ is the binary outcome of the unit $l$ in component $S_{ij}$. It should be pointed that a unit is not necessarily a physical object in a component, but it represents just a Bernoulli observation of the functioning/failure state of that component. Hence, for each subsystem $S_j$, one must allocate

$$T_j = \sum_{i=1}^{n_j} M_{ij} \tag{2.6}$$

units such that the estimated reliability of the system is based on a total sample size:

$$T = \sum_{j=1}^{n} T_j. \tag{2.7}$$

As in the series case, with the help of $s$-independence and the fact that a sample mean is an unbiased estimator of a Bernoulli parameter, the variance of the estimated reliability $\widehat{R}$ incurred by any allocation scheme can be obtained, see [1, 10, 11]:

$$\operatorname{Var}\left\{\widehat{R}\right\} = \prod_{j=1}^{n}\left(\operatorname{Var}\left(\widehat{R}_j\right) + R_j^2\right) - \prod_{j=1}^{n} R_j^2, \tag{2.8}$$

where

$$\operatorname{Var}\left\{\widehat{R}_j\right\} = \left(1 - R_j\right)^2 \left[\prod_{i=1}^{n_j}\left(1 + \frac{c_{ij}^{-2}}{M_{ij}}\right) - 1\right] \tag{2.9}$$

is given as a function of the allocation numbers $M_{ij}$ and the coefficients of variation of Bernoulli populations:

$$c_{ij} = \sqrt{\frac{1}{R_{ij}} - 1}. \tag{2.10}$$

We have found it convenient to work with the equivalent expression of (2.9):

$$\operatorname{Var}\left\{\widehat{R}_j\right\} = \left(1 - R_j\right)^2 \left[\sum_{i=1}^{n_j}\frac{c_{ij}^{-2}}{M_{ij}} + F\left(\frac{c_{1j}^{-2}}{M_{1j}}, \ldots, \frac{c_{n_j j}^{-2}}{M_{n_j j}}\right)\right], \tag{2.11}$$

where

$$F\left(\frac{c_{1j}^{-2}}{M_{1j}}, \ldots, \frac{c_{n_j j}^{-2}}{M_{n_j j}}\right) \tag{2.12}$$

is a sum over all the products of at least two of its arguments.

The problem is to estimate $R$ when components' reliabilities are unknown and a total number of $T$ units must be tested in the system at components' level. The aim is to minimize the variance of $\widehat{R}$. Hence, the problem can be addressed by developing allocation schemes to select $M_{ij}$, the numbers of units to be tested in each component $i$ in the subsystem $j$, under the constraint

$$\sum_{j=1}^{n}\sum_{i=1}^{n_j} M_{ij} = T, \tag{2.13}$$

such that the variance of $\widehat{R}$ is as small as possible.

Reliability sequential schemes (R-SS) exist for the series, parallel, or parallel-series configurations when the sample sizes $T_j$ of the subsystems are fixed, (cf., e.g., [9–11]).

Therefore, one can fully optimize the variance of $\widehat{R}$ just by applying the (R-SS) to find the best partition $T_1, T_2, \ldots, T_n$ of $T$. Unfortunately, a full sequential design cannot be used in practice for large systems since the number of operations will grow dramatically. For this reason, we reasonably propose a hybrid two-stage design which is shown to be asymptotically optimal when $T$ is large.

### *2.1. Lower Bound for the Variance of the Estimated Reliability of the Parallel Subsystem $S_j$*

For the asymptotic optimization of the variance of the estimated reliabilities, we make use of the well-known Lagrange's identity which can be written in the following form.

Let $a_i > 0$, $N_i > 0$, for $i = 1, \ldots, k$ and $N = N_1 + \cdots + N_k$, then the following identity holds:

$$\sum_{i=1}^{k} \frac{a_i}{N_i} = N^{-1} \left[ \left( \sum_{i=1}^{k} \sqrt{a_i} \right)^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \frac{\left(N_i \sqrt{a_j} - N_j \sqrt{a_i}\right)^2}{N_i N_j} \right]. \tag{2.14}$$

**Proposition 2.1.** *Denote by*

$$Q_j = \left(1 - R_j\right)^2 T_j^{-1} \left( \sum_{i=1}^{n_j} c_{ij}^{-1} \right)^2, \tag{2.15}$$

*then*

$$\operatorname{Var}\left\{ \widehat{R}_j \right\} \geq Q_j. \tag{2.16}$$

*Proof.* The proof is a direct consequence of the previous identity (2.14). Indeed,

$$\begin{aligned} \operatorname{Var}\left\{ \widehat{R}_j \right\} = {} & \left(1 - R_j\right)^2 T_j^{-1} \left( \sum_{i=1}^{n_j} c_{ij}^{-1} \right)^2 + T_j^{-1} \left(1 - R_j\right)^2 \sum_{i=1}^{n_j - 1} \sum_{k=i+1}^{n_j} \frac{\left( M_{ij} c_{kj}^{-1} - M_{kj} c_{ij}^{-1} \right)^2}{M_{ij} M_{kj}} \\ & + \left(1 - R_j\right)^2 F\left( \frac{c_{1j}^{-2}}{M_{1j}}, \frac{c_{2j}^{-2}}{M_{2j}}, \ldots, \frac{c_{n_j j}^{-2}}{M_{n_j j}} \right). \end{aligned} \tag{2.17}$$

□

### *2.2. The Two-Stage Design for the Parallel Subsystem $S_j$*

Following the expansion (2.17) and since $F$ contains second-order terms (see later), one gives interest to the numbers $M_{ij}$ which minimize the expression

$$T_j^{-1} \sum_{i=1}^{n_j - 1} \sum_{k=i+1}^{n_j} \frac{\left( M_{ij} c_{kj}^{-1} - M_{kj} c_{ij}^{-1} \right)^2}{M_{ij} M_{kj}}. \tag{2.18}$$

Thus, $M_{ij}$ must verify for $i = 1, \ldots, n_j$:

$$M_{ij} c_{kj}^{-1} = M_{kj} c_{ij}^{-1}, \tag{2.19}$$

which implies that

$$M_{ij} = T_j \frac{c_{ij}^{-1}}{\sum_{k=1}^{n_j} c_{kj}^{-1}}. \tag{2.20}$$

If one assumes that $T_j$ is fixed then a proper two-stage scheme can be used to determine $M_{ij}$, just as was defined in [11], as follows.

Choose $L_j$ as a function of $T_j$ such that

(i) $L_j$ must be large if $T_j$ is large,

(ii) $L_j \leq (T_j / n_j)$,

(iii) $\lim_{T_j \to \infty} (L_j / T_j) = 0$.

One can take for example $L_j = [\sqrt{T_j}]$, where $[\cdot]$ denotes the integer part.

*Stage 1.* Sample $L_j$ units from each component $i$ in the subsystem $j$ estimate $c_{ij}$ by its maximum likelihood estimator (M.L.E)

$$\widehat{c}_{ij} = \sqrt{\frac{L_j}{\sum_{l=1}^{L_j} X_{ij}^{(l)}} - 1} \tag{2.21}$$

and define the predictor, according to (2.20),

$$\widehat{M}_{ij} = \left[ T_j \frac{\widehat{c}_{ij}^{-1}}{\sum_{k=1}^{n_j} \widehat{c}_{kj}^{-1}} \right], \quad i = 1, \ldots, n_j - 1. \tag{2.22}$$

*Stage 2.* Sample $T_j - n_j L_j$ units for which $M_{ij} - L_j$ are units from component $i$ in the subsystem $j$, where $M_{ij}$ is the corrector of $\widehat{M}_{ij}$, defined by

$$\begin{aligned} M_{ij} &= \max\left\{ L_j, \widehat{M}_{ij} \right\}, \quad i = 1, \ldots, n_j - 1, \\ M_{n_j j} &= T_j - \sum_{k=1}^{n_j - 1} M_{kj}. \end{aligned} \tag{2.23}$$

**Theorem 2.2.** *Choosing the $M_{ij}$ according to the previous two-stage sampling scheme, one obtains*

$$\lim_{T_j \to \infty} T_j \left( \operatorname{Var}\left\{ \widehat{R}_j \right\} - Q_j \right) = 0. \tag{2.24}$$

*Proof.* From relation (2.17), one can write

$$T_j\left(\operatorname{Var}\left\{\widehat{R}_j\right\} - Q_j\right) = (1-R_j)^2 \sum_{i=1}^{n_j-1}\sum_{k=i+1}^{n_j} \frac{\left(M_{ij}c_{kj}^{-1} - M_{kj}c_{ij}^{-1}\right)^2}{M_{ij}M_{kj}} + (1-R_j)^2 T_j \cdot F\left(\frac{c_{1j}^{-2}}{M_{1j}},\ldots,\frac{c_{n_jj}^{-2}}{M_{n_jj}}\right). \tag{2.25}$$

When $T_j$ is large enough, condition (iii) gives $M_{ij} = \widehat{M}_{ij}$ for $i = 1,\ldots,n_j - 1$. So the strong law of large numbers with the integer part properties give, when $T_j \to \infty$,

$$\frac{M_{ij}}{M_{kj}} \longrightarrow \frac{c_{kj}}{c_{ij}}, \tag{2.26}$$

for $i = 1,\ldots,n_j$. Hence,

$$\frac{\left(M_{ij}c_{kj}^{-1} - M_{kj}c_{ij}^{-1}\right)^2}{M_{ij}M_{kj}} = \frac{M_{ij}}{M_{kj}}\left(c_{kj}^{-1} - \frac{M_{kj}}{M_{ij}}c_{ij}^{-1}\right)^2 \longrightarrow 0, \quad \text{as } T_j \longrightarrow \infty, \tag{2.27}$$

and on the other hand,

$$T_j \cdot F\left(\frac{c_{1j}^{-2}}{M_{1j}},\ldots,\frac{c_{n_jj}^{-2}}{M_{n_jj}}\right) \longrightarrow 0, \quad \text{as } T_j \longrightarrow \infty, \tag{2.28}$$

which achieves the proof. □

## 3. Lower Bound for the Variance of the Estimated Reliability of the Parallel-Series System

We consider now the parallel-series system $S$. From expression (2.8), one can write

$$\operatorname{Var}\left\{\widehat{R}\right\} = R^2\left[\prod_{j=1}^{n}\left(\frac{\operatorname{Var}\left(\widehat{R}_j\right)}{R_j^2} + 1\right) - 1\right]. \tag{3.1}$$

The following theorem gives a lower bound for the variance of $\widehat{R}$.

**Theorem 3.1.** *Denote by*

$$Q = T^{-1}R^2\left[\sum_{j=1}^{n}\frac{1-R_j}{R_j}\left(\sum_{i=1}^{n_j}c_{ij}^{-1}\right)\right]^2, \tag{3.2}$$

*then*

$$\operatorname{Var}\left\{\widehat{R}\right\} \geq Q. \tag{3.3}$$

*Proof.* Expanding the right-hand side of (2.8) and using (2.1), one obtains

$$\operatorname{Var}\left\{\widehat{R}\right\} = R^2\left[\sum_{j=1}^{n}\frac{\operatorname{Var}\left(\widehat{R}_j\right)}{R_j^2} + F\left(\frac{\operatorname{Var}\left(\widehat{R}_1\right)}{R_1^2},\ldots,\frac{\operatorname{Var}\left(\widehat{R}_n\right)}{R_n^2}\right)\right], \tag{3.4}$$

which gives, with the help of Theorem 2.2,

$$\operatorname{Var}\left\{\widehat{R}\right\} \geq R^2\sum_{j=1}^{n}\frac{Q_j}{R_j^2} = R^2\sum_{j=1}^{n}\frac{\left(((1-R_j)/R_j)\sum_{i=1}^{n_j}c_{ij}^{-1}\right)^2}{T_j}. \tag{3.5}$$

This last expression has the form

$$R^2\sum_{j=1}^{n}\frac{a_j}{T_j}, \tag{3.6}$$

which can be expanded, thanks to identity (2.14), as follows:

$$\begin{aligned} &R^2T^{-1}\left[\sum_{j=1}^{n}\frac{1-R_j}{R_j}\left(\sum_{k=1}^{n_j}c_{kj}^{-1}\right)\right]^2 \\ &\quad + R^2T^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\left(T_i((1-R_j)/R_j)\sum_{k=1}^{n_j}c_{kj}^{-1} - T_j((1-R_i)/R_i)\sum_{k=1}^{n_i}c_{ki}^{-1}\right)^2}{T_iT_j}, \end{aligned} \tag{3.7}$$

and, as a consequence,

$$\operatorname{Var}\left\{\widehat{R}\right\} \geq T^{-1}R^2\left[\sum_{j=1}^{n}\frac{1-R_j}{R_j}\left(\sum_{k=1}^{n_j}c_{kj}^{-1}\right)\right]^2 = Q, \tag{3.8}$$

which achieves the proof. □

## 4. The Hybrid Two-Stage Design for the Parallel-Series System $S$

Similarly to the case of a subsystem and from expressions (3.5) and (3.7), one gives interest to the numbers $T_j$ which minimize the quantity

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\left(T_i((1-R_j)/R_j)\sum_{k=1}^{n_j}c_{kj}^{-1}-T_j((1-R_i)/R_i)\sum_{k=1}^{n_i}c_{ki}^{-1}\right)^2}{T_iT_j} \tag{4.1}$$

and obtain the asymptotic optimality criteria:

$$\frac{T_i}{T_j}=\frac{((1-R_i)/R_i)\sum_{k=1}^{n_i}c_{ki}^{-1}}{((1-R_j)/R_j)\sum_{k=1}^{n_j}c_{kj}^{-1}}, \tag{4.2}$$

for all $i,j\in\{1,2,\dots,n\}$, which gives the rule

$$T_j=T\frac{((1-R_j)/R_j)\sum_{k=1}^{n_j}c_{kj}^{-1}}{\sum_{k=1}^{n}((1-R_k)/R_k)\sum_{i=1}^{n_k}c_{ik}^{-1}}. \tag{4.3}$$

We can now implement a hybrid two-stage design for the determination of the numbers $T_j$ as well as $M_{ij}$ as follows.

*Stage 1.* Choose $L=[\sqrt{T}]$: one applies the two-stage scheme given in Section 2.1 for each subsystem $S_j$ with $T_j=L$ and $L_j=[\sqrt{T_j}]$. Next, obtain the predictor, according to the rule (4.3),

$$\widehat{T}_j=\left[T\frac{\left(\left(1-\widehat{R}_j\right)/\widehat{R}_j\right)\sum_{k=1}^{n_j}\widehat{c}_{kj}^{-1}}{\sum_{k=1}^{n}\left(\left(1-\widehat{R}_k\right)/\widehat{R}_k\right)\sum_{i=1}^{n_k}\widehat{c}_{ik}^{-1}}\right],\quad j=1,\dots,n-1. \tag{4.4}$$

*Stage 2.* Define the corrector

$$\begin{gathered}T_j=\max\left\{L,\widehat{T}_j\right\},\quad j=1,\dots,n-1,\\ T_n=T-\sum_{j=1}^{n-1}T_j,\end{gathered} \tag{4.5}$$

and take back the two-stage scheme for each subsystem $S_j$ to calculate $M_{ij}$ with the sample size equals $T_j$.

Now, the main result of this paper is given by the following theorem.

**Theorem 4.1.** *Choosing $T_j$ and $M_{ij}$ according to the hybrid two-stage design, one obtains*

$$\lim_{T\to\infty} T\left(\operatorname{Var}\left\{\widehat{R}\right\} - Q\right) = 0, \tag{4.6}$$

*where $Q$ is defined in Theorem 3.1.*

*Proof.* The relation (2.25) implies that

$$\operatorname{Var}\left\{\widehat{R}_j\right\} = Q_j + T_j^{-1}(1-R_j)^2 \sum_{i=1}^{n_j-1}\sum_{k=i+1}^{n_j} \frac{\left(M_{ij}c_{kj}^{-1} - M_{kj}c_{ij}^{-1}\right)^2}{M_{ij}M_{kj}} + (1-R_j)^2 F\left(\frac{c_{1j}^{-2}}{M_{1j}},\ldots,\frac{c_{n_j j}^{-2}}{M_{n_j j}}\right). \tag{4.7}$$

As a consequence of the hybrid two-stage design and the strong law of large numbers, $T/T_j$ and $T_j/M_{ij}$ remain bounded for all $i, j$ as $T \to \infty$. It follows that, as $T \to \infty$,

$$\begin{gathered} F\left(\frac{c_{1j}^{-2}}{M_{1j}},\ldots,\frac{c_{n_j j}^{-2}}{M_{n_j j}}\right) = o\left(T^{-1}\right), \\ T_j^{-1}\sum_{i=1}^{n_j-1}\sum_{k=i+1}^{n_j} \frac{\left(M_{ij}c_{kj}^{-1} - M_{kj}c_{ij}^{-1}\right)^2}{M_{ij}M_{kj}} = o\left(T^{-1}\right), \end{gathered} \tag{4.8}$$

thanks to (2.27) and (2.28). Thus,

$$\operatorname{Var}\left\{\widehat{R}_j\right\} = Q_j + o\left(T^{-1}\right), \quad \text{as } T \longrightarrow \infty, \tag{4.9}$$

which implies that

$$\begin{aligned} \prod_{j=1}^{n}\left(\frac{\operatorname{Var}\left\{\widehat{R}_j\right\}}{R_j^2} + 1\right) &= \prod_{j=1}^{n}\left(\frac{Q_j}{R_j^2} + 1 + o\left(T^{-1}\right)\right) \\ &= \prod_{j=1}^{n}\left(\frac{Q_j}{R_j^2} + 1\right) + o\left(T^{-1}\right). \end{aligned} \tag{4.10}$$

As a consequence,

$$\lim_{T\to\infty} T\left(\operatorname{Var}\left\{\widehat{R}\right\} - Q\right) = R^2 \lim_{T\to\infty} T \cdot \left[\prod_{j=1}^{n}\left(\frac{Q_j}{R_j^2} + 1\right) - 1 - Q\right]. \tag{4.11}$$

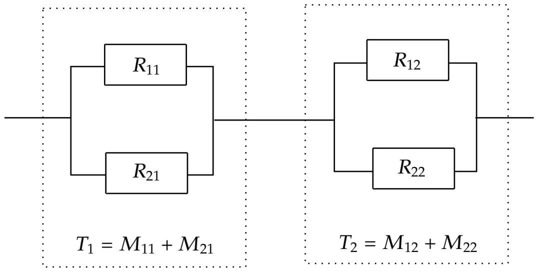


**Figure 1:** A simple parallel-series system of two subsystems with two components each one.

Now, expanding the product within the limit and applying identity (2.14), after having replaced $Q_j$ by its expression (2.15), one obtains

$$\prod_{j=1}^{n}\left(\frac{Q_j}{R_j^2}+1\right)-1 = R^2\left[\sum_{j=1}^{n}\frac{Q_j}{R_j^2}+F\left(\frac{Q_1}{R_1^2},\ldots,\frac{Q_n}{R_n^2}\right)\right] = Q + R^2(A+B), \tag{4.12}$$

where

$$A = T^{-1}\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\frac{\left(T_i((1-R_k)/R_k)\left(\sum_{l=1}^{n_k}c_{lk}^{-1}\right)-T_k((1-R_i)/R_i)\left(\sum_{l=1}^{n_k}c_{li}^{-1}\right)\right)^2}{T_iT_k}, \quad B = F\left(\frac{Q_1}{R_1^2},\ldots,\frac{Q_n}{R_n^2}\right). \tag{4.13}$$

Once more, the hybrid two-stage allocation scheme and the strong law of large numbers provide

$$\lim_{T\to\infty} T\cdot A = 0, \quad \lim_{T\to\infty} T\cdot B = 0, \tag{4.14}$$

which achieves the proof. □

## 5. Monte Carlo Simulation

Let us remark first that the lower bound $Q$ is a first-order approximation of the optimal variance of the reliability estimate under the constraint (2.13) when $T$ is large.

In the first experiment, we will validate the fact that the hybrid scheme provides the best allocation at system level. As in Figure 1, we consider a simple parallel-series system of two subsystems, each one with varying reliabilities and a fixed sample size $T = 20$. For each

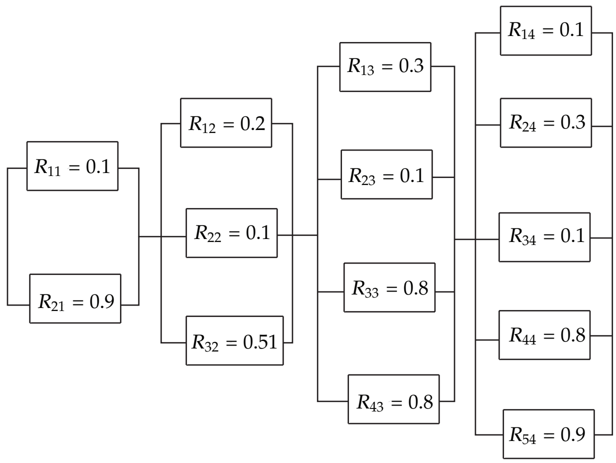


**Figure 2:** A nontrivial parallel-series system.

situation A, B, C, and D and for each partition sample size $\{T_1, T - T_1\}$ where $T_1$ varies from $[\sqrt{T}]$ to $T - [\sqrt{T}]$, by step of 1, we have applied the proper two-stage design for each parallel subsystem and reported in a bar diagram $\mathrm{Var}(\widehat{R})$ as a function of $T_1$, see Figure 3. On the other hand, in Table 1, we have reported the expected value of $T_1 = M_{11} + M_{21}$ given by the hybrid two-stage design. As expected, our scheme gives the best allocation for each situation.

The second experiment deals with a nontrivial parallel-series system just as in [11], where subsystems are composed, respectively, of 2, 3, 4, and 5 components, see Figure 2. The partition total numbers $T_j$ to test in each subsystem are evaluated systematically by the hybrid two-stage design while their sum $T$ is incremented from 100 to 10000 by step of 100. Figure 4 shows the rate of the excess of variance $T(\mathrm{Var}(\widehat{R}) - Q)$ at logarithmic scale as a function of the sample size $T$. The asymptotic optimality of the hybrid scheme is validated.

## 6. Conclusion

The proof of the first-order asymptotic optimality for the proper two-stage design for a parallel subsystem as well as for the hybrid two-stage design for the full system has been obtained mainly through the following steps:

(i) an adequate writing of the variance of the reliability estimate;

(ii) a lower bound for this variance, independent of allocation;

(iii) the allocation defined by the hybrid sampling scheme and the strong law of large numbers.

With a straightforward but tedious adaptation, the above study can be, namely, extended to deal with complex systems involving a multicriteria optimization problem under a set of constraints such as risk, system weight, cost, and performance, in a fixed or in a Bayesian framework.

An interesting feature of the asymptotic optimality in reliability allocation problems is to consider second-order procedures which, to our knowledge, is not a trivial task and needs

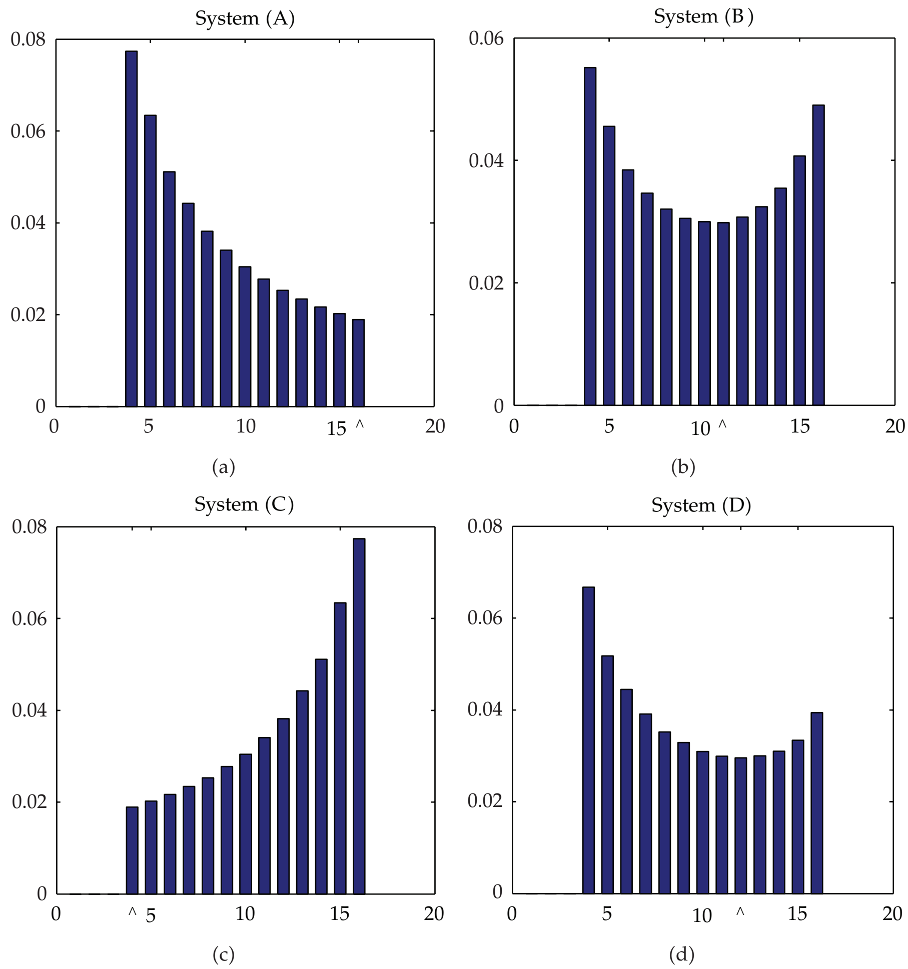


**Figure 3:** Bar diagram $\mathrm{Var}(\widehat{R})$ as a function of $T_1$ for each case A, B, C, and D $\hat{}$ shows the minimum of $\mathrm{Var}(\widehat{R})$.

**Table 1:** Expected value of $T_1 = M_{11} + M_{21}$ given by the hybrid two-stage design.

| System | $R_{11}$ | $R_{21}$ | $R_{12}$ | $R_{22}$ | $E(T_1)$ |
|---|---|---|---|---|---|
| A | 0.1 | 0.11 | 0.9 | 0.99 | 16 |
| B | 0.5 | 0.55 | 0.51 | 0.6 | 11 |
| C | 0.9 | 0.99 | 0.1 | 0.11 | 4 |
| D | 0.2 | 0.4 | 0.6 | 0.3 | 12 |

more investigation in the construction of the lower bound of the variance of the reliability estimate and more understanding of the rate of convergence in the strong law of large numbers.

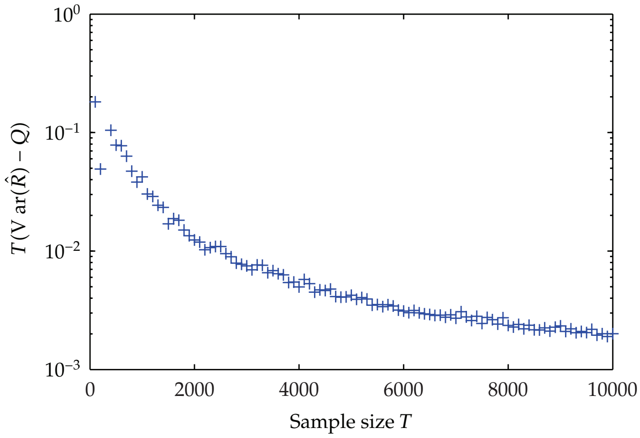


**Figure 4:** Asymptotic optimality of the hybrid two-stage design: the speed of the excess of variance $T(\mathrm{Var}(\widehat{R}) - Q)$ at logarithmic scale as a function of the sample size $T$.

## Acknowledgments

The authors would like to thank the referees for their helpful comments on the last version of the draft. This work is supported by grants from the National Research Project (PNR) and the L.A.A.R Laboratory of the Department of Physics at Mohamed Boudiaf University of Oran.

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