Research Article

Uniform Estimate of the Finite-Time Ruin Probability for All Times in a Generalized Compound Renewal Risk Model

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We discuss the uniformly asymptotic estimate of the finite-time ruin probability for all times in a generalized compound renewal risk model, where the interarrival times of successive accidents and all the claim sizes caused by an accident are two sequences of random variables following a wide dependence structure. This wide dependence structure allows random variables to be either negatively dependent or positively dependent.

1. Introduction

In this section, we will introduce a generalized compound renewal risk model, some common classes of heavy-tailed distributions, and some dependence structures of random variables (r.v.s), respectively.

1.1. Risk Model

It is well known that the compound renewal risk model was first introduced by Tang et al. [1], and since then it has been extensively investigated by many researchers, for example, Aleškevičienė et al. [2], Zhang et al. [3], Lin and Shen [4], Yang et al. [5], and the references therein. In the paper, we consider a generalized compound renewal risk model which satisfies the following assumptions.

Assumption H_1

The interarrival times $\{\theta_i, i \geq 1\}$ of successive accidents are nonnegative, identically distributed, but not necessarily independent r.v.s with finite mean λ^{-1}.

Assumption H_2

The claim sizes and their number caused by nth accident are $\{X_i^{(n)}, i \geq 1\}$ and N_n, $n \geq 1$, respectively, where $\{X_i^{(n)}, i \geq 1, n \geq 1\}$ are nonnegative and identically distributed r.v.s with common distribution F and finite mean μ, and $\{X_i^{(n)}, i \geq 1\}$ are not necessarily independent r.v.s, but $\{X_i^{(n)}, i \geq 1\}$ and $\{X_i^{(m)}, i \geq 1\}$ are mutually independent for all $n \neq m$, $n, m \geq 1$, while $\{N_n, n \geq 1\}$ are independent, identically distributed (i.i.d.), and positive integer-valued r.v.s with common distribution G and finite mean ν.

Assumption H_3

The sequences $\{\theta_i, i \geq 1\}$, $\{X_i^{(n)}, i \geq 1, n \geq 1\}$, and $\{N_n, n \geq 1\}$ are mutually independent.
    Denote the arrival times of the nth accident by $\tau_n = \sum_{i=1}^n \theta_i$, $n \geq 1$, which can form a nonstandard renewal counting process

$$N(t) = \sup\{n \geq 1, \tau_n \leq t\}, \quad t \geq 0, \tag{1.1}$$

with mean function $\lambda(t) = EN(t)$. Hence the total claim amount at time τ_n and the total claim amount up to time $t \geq 0$ are, respectively,

$$S_{N_n}^{(n)} = \sum_{i=1}^{N_n} X_i^{(n)}, \qquad S(t) = \sum_{n=1}^{N(t)} S_{N_n}^{(n)}, \tag{1.2}$$

and then the insurer's surplus process is given by

$$R(t) = x + ct - S(t), \quad t \geq 0, \tag{1.3}$$

where $x \geq 0$ is the initial surplus and $c > 0$ is the constant premium rate. The finite-time ruin probability within time $t > 0$ is defined as

$$\psi(x, t) = P\left(\inf_{0 \leq s \leq t} R(s) < 0 \mid R(0) = x\right). \tag{1.4}$$

Clearly, the ruin can only arise at the times τ_n, $1 \leq n \leq N(t)$, then

$$\psi(x, t) = P\left(\max_{0 \leq k \leq N(t)} \sum_{n=1}^k \left(S_{N_n}^{(n)} - c\theta_n\right) > x\right). \tag{1.5}$$

Let τ be a nonnegative r.v., the random time ruin probability is

$$\psi(x, \tau) = P\left(\max_{0 \leq k \leq N(\tau)} \sum_{n=1}^{k} \left(S_{N_n}^{(n)} - c\theta_n\right) > x\right).$$ (1.6)

In order for the ultimate ruin not to be certain, we assume the safety loading condition holds, namely,

$$\kappa = c\lambda^{-1} - \nu\mu > 0.$$ (1.7)

In the generalized compound renewal risk model above, if all the sequences $\{\theta_i, i \geq 1\}$, $\{X_i^{(n)}, i \geq 1, n \geq 1\}$, and $\{N_n, n \geq 1\}$ are i.i.d. r.v.s, then the model is reduced to the standard compound renewal risk model introduced by Tang et al. [1], if $N_1 = N_2 = \cdots = 1$, then the model is the renewal risk model, see Tang [6], Leipus and Šiaulys [7], Yang et al. [8], and Wang et al. [9], among others.

1.2. Heavy-Tailed Distribution Classes

We now present some common classes of heavy-tailed distributions. Firstly, we introduce some notions and notation. All limit relationships in the paper are for $x \rightarrow \infty$ unless mentioned otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$, write $a(x) \sim b(x)$ if both, write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$. For two positive bivariate functions $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, we say that relation $a(x, t) \sim b(x, t)$ holds uniformly for all $t \in \Delta \neq \emptyset$ if

$$\lim_{x \to \infty} \sup_{t \in \Delta} \left| \frac{a(x, t)}{b(x, t)} - 1 \right| = 0.$$ (1.8)

For a distribution V on $(-\infty, \infty)$, denote its tail by $\overline{V}(x) = 1 - V(x)$, and its upper and lower Matuszewska indices by, respectively, for $y > 1$,

$$J_V^+ = -\lim_{y \to \infty} \frac{\log \overline{V}_*(y)}{\log y}, \qquad J_V^- = -\lim_{y \to \infty} \frac{\log \overline{V}^*(y)}{\log y},$$ (1.9)

where $\overline{V}_*(y) = \liminf \overline{V}(xy)/\overline{V}(x)$ and $\overline{V}^*(y) = \limsup \overline{V}(xy)/\overline{V}(x)$.

Chistyakov [10] introduced an important class of heavy-tailed distributions, the subexponential class. By definition, a distribution V on $[0, \infty)$ belongs to the subexponential class, denoted by $V \in \mathcal{S}$, if

$$\overline{V^{*2}}(x) \sim 2\overline{V}(x),$$ (1.10)

where V^{*2} denotes the 2-fold convolution of V. Clearly, if $V \in \mathcal{S}$ then V is long tailed, denoted by $V \in \mathcal{L}$ and characterized by

$$\overline{V}(x + y) \sim \overline{V}(x), \quad \forall y > 0. \tag{1.11}$$

One can easily see that a distribution $V \in \mathcal{L}$ if and only if there exists a function $f(\cdot) : [0, \infty) \mapsto [0, \infty)$ such that

$$f(x) \uparrow \infty, \quad f(x) = o(x), \quad \overline{V}(x \pm f(x)) \sim \overline{V}(x). \tag{1.12}$$

Korshunov [11] introduced a subclass of the class $\mathcal{S}$, the strongly subexponential class, denoted by $\mathcal{S}_*$. Say that a distribution $V \in \mathcal{S}_*$, if $\int_0^\infty \overline{V}(y) dy < \infty$ and

$$\overline{V_u^{*2}}(x) \sim 2\overline{V_u}(x) \tag{1.13}$$

holds uniformly for $u \in [1, \infty)$, where $\overline{V_u}(x) = \min\{1, \int_x^{x+u} \overline{V}(y) dy\} \mathbf{1}_{\{x \geq 0\}} + \mathbf{1}_{\{x < 0\}}$ with $\mathbf{1}_A$ an indicator function of set A. Feller [12] introduced another important class of heavy-tailed distributions, the dominant variation class, which is not mutually inclusive with the class $\mathcal{L}$. Say that a distribution V on $[0, \infty)$ belongs to the dominant variation class, denoted by $V \in \mathfrak{D}$, if

$$\overline{V}^*(y) < \infty, \quad \forall y > 0. \tag{1.14}$$

Cline [13] introduced a slightly smaller class of $\mathcal{L} \cap \mathfrak{D}$, the consistent variation class, denoted by $\mathcal{C}$. Say that a distribution $V \in \mathcal{C}$ if

$$\lim_{y \searrow 1} \overline{V}_*(y) = 1, \quad \text{or equivalently,} \quad \lim_{y \nearrow 1} \overline{V}^*(y) = 1. \tag{1.15}$$

Specially, the class $\mathcal{C}$ covers a famous class $\mathcal{R}$, called the regular variation class. By definition, a distribution $V \in \mathcal{R}_{-\alpha}$, if there exists some $\alpha > 0$ such that

$$\lim \frac{\overline{V}(xy)}{\overline{V}(x)} = y^{-\alpha}, \quad \forall y > 0. \tag{1.16}$$

It is well known that for the distributions with finite mean, the following inclusion relationships hold properly, namely,

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathfrak{D} \subset \mathcal{S}_* \subset \mathcal{S} \subset \mathcal{L}, \tag{1.17}$$

see, for example, Cline and Samorodnitsky [14], Klüppelberg [15], Embrechts et al. [16], and Denisov et al. [17]. For more details of heavy-tailed distributions and their applications to finance and insurance, the readers are referred to Bingham et al. [18] and Embrechts et al. [16].

1.3. Wide Dependence Structure

In this section we will introduce some concepts and properties of a wide dependence structures of r.v.s, which was first introduced by Wang et al. [19] as follows.

Definition 1.1. Say that r.v.s $\{\xi_i, i \geq 1\}$ are widely upper orthant dependent (WUOD), if for each $n \geq 1$, there exists some finite positive number $g_U(n)$ such that, for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n}\{\xi_i > x_i\}\right) \leq g_U(n)\prod_{i=1}^{n}P(\xi_i > x_i). \tag{1.18}$$

Say that r.v.s $\{\xi_i, i \geq 1\}$ are widely lower orthant dependent (WLOD), if for each $n \geq 1$, there exists some finite positive number $g_L(n)$ such that, for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n}\{\xi_i \leq x_i\}\right) \leq g_L(n)\prod_{i=1}^{n}P(\xi_i \leq x_i). \tag{1.19}$$

Furthermore, $\{\xi_i, i \geq 1\}$ are said to be widely orthant dependent (WOD) if they are both WUOD and WLOD.

The WUOD, WLOD, and WOD r.v.s are collectively called as widely dependent r.v.s. Recall that if $g_U(n) \equiv 1$ or $g_L(n) \equiv 1$ for each $n \geq 1$ in Definition 1.1, then $\{X_i, i \geq 1\}$ are negatively upper orthant dependent or negatively lower orthant dependent (NLOD), see Ebrahimi and Ghosh [20] or Block et al. [21]; if $g_U(n) = g_L(n) \equiv M$ for some constant $M > 0$ and each $n \geq 1$ such that the two inequalities in Definition 1.1 both hold, then $\{X_i, i \geq 1\}$ are extended negatively dependent, see Liu [22] and Chen et al. [23]. Obviously, the WUOD and WLOD structures allow a wide range of negative dependence structures among r.v.s, such as extended negative dependence, negatively upper orthant dependence/negatively lower orthant dependence, negative association (see Joag-Dev and Proschan [24]), and even some positive dependence. For some examples to illustrate that the WUOD and WLOD structures allow some negatively and positively dependent r.v.s, we refer the readers to Wang et al. [19].

The following properties for widely dependent r.v.s can be obtained immediately below.

Proposition 1.2. (1) *Let $\{\xi_i, i \geq 1\}$ be WLOD (or WUOD). If $\{f_i(\cdot), i \geq 1\}$ are nondecreasing, then $\{f_i(\xi_i), i \geq 1\}$ are still WLOD (or WUOD); if $\{f_i(\cdot), i \geq 1\}$ are nonincreasing, then $\{f_i(\xi_i), i \geq 1\}$ are WUOD (or WLOD).*

(2) *If $\{\xi_i, i \geq 1\}$ are nonnegative and WUOD, then for each $n \geq 1$,*

$$E\prod_{i=1}^{n}\xi_i \leq g_U(n)\prod_{i=1}^{n}E\xi_i. \tag{1.20}$$

Particularly, if $\{\xi_i, i \geq 1\}$ are WUOD, then for each $n \geq 1$ and any $s > 0$,

$$E \exp\left\{ s \sum_{i=1}^{n} \xi_i \right\} \leq g_U(n) \prod_{i=1}^{n} E \exp\{s\xi_i\}. \tag{1.21}$$

Following the wide dependence structures as above, we will consider a generalized compound renewal risk model satisfying *Assumption H_3* and the following specific assumptions.

Assumption H_1^*

The interarrival times $\{\theta_i, i \geq 1\}$ are nonnegative, identically distributed and WLOD r.v.s with finite mean λ^{-1}.

Assumption H_2^*

The claim sizes caused by nth accident $\{X_i^{(n)}, i \geq 1\}$ are nonnegative, identically distributed and WUOD r.v.s, and the other statements of *Assumption H_2* are still valid.

The rest of this work is organized as follows: in Section 2 we will state the motivations and main results of this paper after presenting some existing results, and in Section 3 we will give some lemmas and then prove the main results.

2. Main Results

In this section, we will present our main results of this paper. Before this, we prepare some related results and the motivations of the main results. For later use, we define $\Lambda = \{t : \lambda(t) > 0\} = \{t : P(\tau_1 \leq t) > 0\}$.

2.1. Related Results and Motivations

As mentioned above, the asymptotics for the finite-time ruin probability in the compound renewal risk model have been studied by many authors. Among them, Aleškevičienė et al. [2] considered the standard compound renewal risk model with condition (1.7) and showed that

(i) if $G \in \mathcal{C}$, $\overline{F}(x) = o(\overline{G}(x))$ and $E\theta_1^p < \infty$ for some $p > J_G^+ + 1$, then it holds uniformly for all $t \in \Lambda$ that

$$\psi(x, t) \sim \frac{1}{\kappa} \int_x^{x+\kappa\lambda(t)} G\left(\frac{s}{\mu}\right) ds; \tag{2.1}$$

(ii) if $F \in \mathcal{C}$, $\overline{G}(x) = o(\overline{F}(x))$ and $E\theta_1^p < \infty$ for some $p > J_F^+ + 1$, then it holds uniformly for all $t \in \Lambda$ that

$$\psi(x, t) \sim \frac{\nu}{\kappa} \int_x^{x+\kappa\lambda(t)} F(s) ds; \tag{2.2}$$

(iii) if $F \in \mathcal{R}_{-\alpha}$, $\overline{G}(x) \sim c\overline{F}(x)$ for some constant $c > 0$, and $E\theta_1^p < \infty$ for some $p > \alpha + 1$, then it holds uniformly for all $t \in \Lambda$ that

$$\psi(x,t) \sim \frac{\nu + c\mu^\alpha}{\kappa} \int_x^{x+\kappa\lambda(t)} \overline{F}(s)ds. \tag{2.3}$$

Recently, Zhang et al. [3] extended the results of Aleškevičienė et al. [2] to the case that the claim sizes $\{X_i^{(n)}, i \geq 1\}$ caused by nth accident are negatively associated and obtained a unified form of $\psi(x,t)$ as follows: let $\overline{F}(x) \sim c\overline{G}(x)$ for $c \in [0, \infty]$, and one of the conditions below holds: (i) for $c = 0$, $G \in \mathcal{C}$, and $E\theta_1^p < \infty$ for some $p > J_G^+ + 1$; (ii) for $c > 0$, $F \in \mathcal{C}$, and $E\theta_1^p < \infty$ for some $p > J_F^+ + 1$; then it holds uniformly for all $t \in \Lambda$ that

$$\psi(x,t) \sim \frac{1}{\kappa} \int_x^{x+\kappa\lambda(t)} \left(\nu \overline{F}(s) + \overline{G}\left(\frac{s}{\mu}\right) \right) ds. \tag{2.4}$$

Observe the results of Zhang et al. [3] especially extended case (iii) of Aleškevičienė et al. [2]. Also, Lin and Shen [4] considered a generalized compound renewal risk model with $\{X_i^{(n)}, i \geq 1\}$ satisfying one type of asymptotically quadrant subindependent structure and also obtained the same relations (2.2), (2.3), and (2.4) as that of Aleškevičienė et al. [2].

Inspired by the above results, we will further discuss some issues as follows:

(1) to cancel the moment condition on $\{\theta_i, i \geq 1\}$, namely, $E\theta_1^p < \infty$ for some $p > J_G^+ + 1$, and $E\theta_1^p < \infty$ for some $p > J_F^+ + 1$;

(2) to extend partially the class $\mathcal{C}$ or $\mathcal{R}$ to the class $\mathcal{L} \cap \mathcal{D}$;

(3) to discuss the case when $\{X_i^{(n)}, i \geq 1\}$ are WUOD and $\{\theta_i, i \geq 1\}$ are WLOD;

(4) to drop the interrelationships between $\overline{F}$ and $\overline{G}$ and investigate the case when both F and G are heavy tailed.

In the paper, we will answer the four issues directly, and then we obtain our main results in the next section.

2.2. Main Results

For the main results of this paper, we now state some conditions which are that of Wang et al. [9].

Condition 1. The interarrival times $\{\theta_i, i \geq 1\}$ are NLOD r.v.s.

Condition 2. The interarrival times $\{\theta_i, i \geq 1\}$ are WOD r.v.s and there exists a positive and nondecreasing function $g(x)$ such that $g(x) \uparrow \infty$, $x^{-k}g(x)\overline{\downarrow}$ for some $0 < k < 1$, $E\theta_1 g(\theta_1) < \infty$, and $\max\{g_U(n), g_L(n)\} \leq g(n)$ for all $n \geq 1$, where $x^{-k}g(x)\overline{\downarrow}$ means that $x_1^{-k}g(x_1) \geq Cx_2^{-k}g(x_2)$ for all $0 \leq x_1 < x_2 < \infty$ and some finite constant $C > 0$.

Condition 3. The interarrival times $\{\theta_i, i \geq 1\}$ are WOD r.v.s with $E\theta_1^p < \infty$ for some $2 \leq p < \infty$ and there exists a constant $\alpha > 0$ such that $\lim_{n \to \infty} \max\{g_U(n), g_L(n)\}n^{-\alpha} = 0$.

Condition 4. The interarrival times $\{\theta_i, i \geq 1\}$ are WOD r.v.s with $Ee^{\beta\theta_1} < \infty$ for some $0 < \beta < \infty$ and $\lim_{n\to\infty} \max\{g_U(n), g_L(n)\}e^{-\gamma n} = 0$ for any $\gamma > 0$.

The first main result of this paper is the following.

Theorem 2.1. *Consider the generalized compound renewal risk model with Assumptions H_1^*, H_2^*, and H_3 and condition (1.7), there exists a finite constant $\alpha > 0$ such that*

$$\lim_{n\to\infty} g_L(n)n^{-\alpha} = 0, \qquad \lim_{n\to\infty} g_U(n)e^{-\gamma n} = 0, \quad for \ any \ \gamma > 0. \tag{2.5}$$

Meanwhile, let one of Conditions 1–4 hold, and one of Conditions 1–4 with $\{\theta_i, i \geq 1\}$ replaced by $\{X_i^{(n)}, i \geq 1\}$ still holds.

(1) *If $\overline{F}(x) \sim c\overline{G}(x)$ for some $c \in [0, \infty]$, additionally, for $c = 0$ and $G \in \mathcal{C}$; for $0 < c < \infty$ and $F \in \mathcal{C}$; for $c = \infty$ and $F \in \mathcal{L} \cap \mathcal{D}$, then relation (2.4) holds uniformly for all $t \in \Lambda$.*

(2) *If $F \in \mathcal{L} \cap \mathcal{D}$ and $G \in \mathcal{C}$, relation (2.4) holds uniformly for all $t \in \Lambda$.*

Note that there do exist some WLOD r.v.s satisfying condition (2.5), see Wang et al. [19]. In the second main result below, we discuss the random time ruin probability, which requires another assumption.

Assumption H_4

Let τ be nonnegative r.v. and independent of the sequences $\{\theta_i, i \geq 1\}$, $\{X_i^n, i \geq 1, n \geq 1\}$, and $\{N_n, n \geq 1\}$.

Define $\Delta = \{\tau : P(\tau \in \Lambda) > 0\}$.

Theorem 2.2. *Under conditions of Theorem 2.1 and Assumption H_4, one has*

(1) *if $\overline{F}(x) \sim c\overline{G}(x)$ for some $c \in [0, \infty]$, additionally, for $c = 0$ and $G \in \mathcal{C}$; for $0 < c < \infty$ and $F \in \mathcal{C}$; for $c = \infty$ and $F \in \mathcal{L} \cap \mathcal{D}$, then it holds uniformly for all $t \in \Delta$ that*

$$\psi(x, \tau) \sim \frac{1}{\kappa}E \int_x^{x+\kappa\lambda(\tau)} \left(\nu F(s) + \overline{G}\left(\frac{s}{\mu}\right)\right) ds; \tag{2.6}$$

(2) *if $F \in \mathcal{L} \cap \mathcal{D}$ and $G \in \mathcal{C}$, then relation (2.6) still holds uniformly for all $t \in \Delta$.*

Remark 2.3. According to the proofs below of Theorems 2.1 and 2.2, we can see that Conditions 1–4 are doing nothing more than making $\{\theta_i, i \geq 1\}$ and $\{X_i^n, i \geq 1\}$ satisfy the strong law of large number, namely,

$$\lim_{k\to\infty} k^{-1}\sum_{i=1}^k \theta_i = \lambda^{-1}, \qquad \lim_{k\to\infty} k^{-1}\sum_{i=1}^k X_i^{(n)} = \mu \quad \text{a.s..} \tag{2.7}$$

So, Conditions 1–4 in Theorems 2.1 and 2.2 can be replaced by (2.7).

3. Proofs of Main Results

In this section we will give the proofs of our main results, for which we need some following lemmas.

Lemma 3.1. *If $\{\xi_i, 1 \le i \le n\}$ are n WUOD and nonnegative r.v.s with distributions $V_i \in \mathcal{L} \cap \mathcal{D}$, $1 \le i \le n$, respectively, then for any fixed $0 < a \le b < \infty$,*

$$P\left(\sum_{i=1}^{n} c_i \xi_i > x\right) \sim \sum_{i=1}^{n} P(c_i \xi_i > x) \tag{3.1}$$

holds uniformly for all $\underline{c_n} = (c_1, c_2, \ldots, c_n) \in [a, b]^n$.

Proof. See Lemma 3.1 of Gao et al. [25].

Particularly, let $c_1 = c_2 = \cdots c_n = 1$ in Lemma 3.2, we have a lemma below. □

Lemma 3.2. *If $\{\xi_i, 1 \le i \le n\}$ are n WUOD and nonnegative r.v.s with distributions $V_i \in \mathcal{L} \cap \mathcal{D}$, $1 \le i \le n$, then*

$$P\left(\sum_{i=1}^{n} \xi_i > x\right) \sim \sum_{i=1}^{n} \overline{V}_i(x). \tag{3.2}$$

Lemma 3.3. *If $\{\xi_i, i \ge 1\}$ are WUOD and real-valued r.v.s with common distribution $V \in \mathcal{D}$ and mean 0 and satisfying*

$$\lim_{n \to \infty} g_U(n) e^{-\gamma n} = 0 \quad for \ any \ \gamma > 0, \tag{3.3}$$

then for any $\gamma > 0$, there exists a constants $C = C(\gamma)$ such that

$$P\left(\sum_{i=1}^{n} \xi_i > x\right) \le Cn\overline{V}(x) \tag{3.4}$$

holds for all $x \ge \gamma n$ and all $n \ge 1$.

Proof. By Proposition 1.2 and along the same lines of the proof of Theorem 3.1 of Tang [26] with slight modifications, we can derive that, for some positive integer m,

$$\sup_{n \ge m, x \ge \gamma n} \frac{P\left(\sum_{i=1}^{n} \xi_i > x\right)}{n\overline{V}(x)} < \infty. \tag{3.5}$$

From Lemma 3.2, we have

$$\sup_{1\leq n\leq m, x\geq \gamma n} \frac{P(\sum_{i=1}^{n}\xi_i > x)}{n\overline{V}(x)} \leq \sum_{n=1}^{m} \sup_{x\geq \gamma n} \frac{P(\sum_{i=1}^{n}\xi_i > x)}{n\overline{V}(x)} < \infty. \tag{3.6}$$

Combining (3.5) and (3.6), there exists a constant $C(\gamma) > 0$ such that (3.4) holds for all $x \geq \gamma n$ and all $n \geq 1$.   □

The following lemma discusses the strong law of large numbers for widely dependent r.v.s, which is due to Wang and Cheng [27].

Lemma 3.4. *Let $\{\xi_i, i \geq 1\}$ be a sequence of real-valued r.v.s with finite mean $a > 0$ and satisfy one of the Conditions 1–4 with $\{\theta_i, i \geq 1\}$ replaced by $\{\xi_i, i \geq 1\}$. Then*

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \xi_i = a \quad a.s.. \tag{3.7}$$

Proof. Follow Theorem 1 of Matula [28], Theorem 1.4, and the proofs of Theorems 1.1 and 1.2 of Wang and Cheng [27], respectively.   □

The lemma below gives the tail behavior of random sum, which extends the results of Aleškevičienė et al. [2] and Zhang et al. [3].

Lemma 3.5. *Let $\{\xi_i, i \geq 1\}$ be a sequence of identically distributed and real-valued r.v.s with distribution V and finite mean a and satisfy one of the Conditions 1–4 with $\{\theta_i, i \geq 1\}$ replaced by $\{\xi_i, i \geq 1\}$, where for Conditions 2 and 3 one further assumes that (3.3) holds. Let η be nonnegative integer-valued r.v. with distribution U and finite mean b, independent of $\{\xi_i, i \geq 1\}$. Assume that $\overline{V}(x) \sim c\overline{U}(x)$ for some $c \in [0, \infty]$.*

(1) *If $0 \leq c < \infty$, $U \in \mathcal{C}$, and the conditions of Lemma 3.4 are valid, then*

$$P\left(\sum_{i=1}^{\eta} \xi_i > x \right) \sim b\overline{V}(x) + \overline{U}\left(\frac{x}{a}\right). \tag{3.8}$$

(2) *If $c = \infty$ and $V \in \mathcal{L} \cap \mathcal{D}$, then relation (3.8) holds.*

(3) *Let no assumption be made on the interrelationship between $\overline{V}$ and $\overline{U}$. If $V \in \mathcal{L} \cap \mathcal{D}$, $U \in \mathcal{C}$, and the conditions of Lemma 3.4 are still valid, then relation (3.8) still holds.*

Proof. Because η has finite mean, there exists a large integer $m_0 > 0$ such that, for any fixed $\varepsilon > 0$, it holds that

$$E\eta\mathbf{1}_{\{\eta > m_0\}} \leq \varepsilon. \tag{3.9}$$

(1) First consider the case that $0 < c < \infty$. Clearly, $U \in \mathcal{C}$ implies $V \in \mathcal{C}$. For any $x > 0$ and any $\delta \in (0, 1)$, we have

$$
\begin{aligned}
P\left(\sum_{i=1}^{\eta} \xi_i > x\right) &= \sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} \xi_i > x\right) P(\eta = n) \\
&= \left(\sum_{n=1}^{m_0} + \sum_{n=m_0+1}^{(1-\delta)x/a} + \sum_{n>(1-\delta)x/a}\right) P\left(\sum_{i=1}^{n} \xi_i > x\right) P(\eta = n) \\
&= K_1 + K_2 + K_3.
\end{aligned}
\tag{3.10}
$$

For K_1, by Lemma 3.2 it follows that

$$
K_1 \sim \overline{V}(x) \sum_{n=1}^{m_0} n P(\eta = n) \leq b\overline{V}(x).
\tag{3.11}
$$

For K_2, since $n < (1 - \delta)x/a$ and $V \in \mathcal{C} \subset \mathfrak{D}$, we obtain by Lemma 3.3 that

$$
\begin{aligned}
P\left(\sum_{i=1}^{n} \xi_i > x\right) &= P\left(\sum_{i=1}^{n} (\xi_i - a) > x - na\right) \\
&\leq P\left(\sum_{i=1}^{n} (\xi_i - a) > \delta x\right) \\
&\leq C(\gamma) n \overline{V}(\delta x) \\
&\leq \widetilde{C}(\gamma) n \overline{V}(x),
\end{aligned}
\tag{3.12}
$$

where $\gamma = \delta a/(1-\delta)$, $C(\gamma)$ and $\widetilde{C}(\gamma)$ are two constants only depending on γ. Hence, applying (3.9), Lemma 3.2, and the dominated convergence theorem can yield that

$$
K_2 \sim \overline{V}(x) \sum_{n=m_0+1}^{(1-\delta)x/a} n P(\eta = n) \leq \overline{V}(x) E\eta \mathbf{1}_{\{\eta > m_0\}} \leq \varepsilon \overline{V}(x).
\tag{3.13}
$$

For K_3, since $\delta \in (0, 1)$ can be arbitrarily close to 0, we see by $U \in \mathcal{C}$ that

$$
K_3 \leq \overline{U}\left(\frac{(1-\delta)x}{a}\right) \lesssim \overline{U}\left(\frac{x}{a}\right).
\tag{3.14}
$$

Substituting (3.11)–(3.14) into (3.10) and considering the arbitrariness of $\varepsilon > 0$, we derive that

$$
P\left(\sum_{i=1}^{\eta} \xi_i > x\right) \lesssim b\overline{V}(x) + \overline{U}\left(\frac{x}{a}\right).
\tag{3.15}
$$

On the other hand, we note that

$$
\begin{aligned}
P\left(\sum_{i=1}^{\eta}\xi_i > x\right) &= \sum_{n=1}^{\infty}P\left(\sum_{i=1}^{n}\xi_i > x\right)P(\eta = n) \\
&\geq \left(\sum_{n=1}^{m_0} + \sum_{n>(1+\delta)x/a}\right)P\left(\sum_{i=1}^{n}\xi_i > x\right)P(\eta = n) \\
&= K_1 + K_4.
\end{aligned}
\tag{3.16}
$$

For K_1, by Lemma 3.2 and (3.9), we get

$$
K_1 \sim \left(b - E\eta\mathbf{1}_{\{\eta>m_0\}}\right)\overline{V}(x) \geq (b - \varepsilon)\overline{V}(x).
\tag{3.17}
$$

For K_4, by Lemma 3.4 we find that

$$
\lim_{n\to\infty}P\left(\frac{\sum_{i=1}^{n}\xi_i}{n} - a > -\frac{a\delta}{1+\delta}\right) = 1,
\tag{3.18}
$$

which, along with $V \in \mathcal{C}$ and the arbitrariness of $\delta \in (0,1)$, leads to

$$
\begin{aligned}
K_4 &\geq \sum_{n>(1+\delta)x/a}P\left(\frac{\sum_{i=1}^{n}\xi_i}{n} - a > -\frac{a\delta}{1+\delta}\right)P(\eta = n) \\
&\gtrsim \overline{U}\left(\frac{(1+\delta)x}{a}\right) \gtrsim \overline{U}\left(\frac{x}{a}\right).
\end{aligned}
\tag{3.19}
$$

Hence, from (3.16)–(3.19) and the arbitrariness of $\varepsilon > 0$, we obtain that

$$
P\left(\sum_{i=1}^{\eta}\xi_i > x\right) \gtrsim b\overline{V}(x) + \overline{U}\left(\frac{x}{a}\right).
\tag{3.20}
$$

So, combining (3.15) and (3.20) proves that (3.8) holds for $0 < c < \infty$.

Next we turn to the case that $c = 0$, namely, $\overline{V}(x) = o(\overline{U}(x))$. According to Lemma 4.4 of Faÿ et al. [29], there exists a nondecreasing slowly varying function $L(x) \to \infty$ such that $\overline{V}(x) = o(\overline{U}(x)/L(x))$, which results in that for some $x_0 > 0$,

$$
\overline{V}(x) \leq \overline{U}(x)/L(x) \leq 1, \quad \text{for all } x \geq x_0.
\tag{3.21}
$$

Define

$$\overline{V}_*(x) = \begin{cases} 1, & \text{if } 0 \leq x < x_0, \\ \dfrac{\overline{U}(x)}{L(x)}, & \text{if } x \geq x_0, \end{cases} \tag{3.22}$$

$$V_*^{-1}(y) = \inf\{x \in (-\infty, \infty) : V_*(x) \geq y\}, \quad 0 \leq y \leq 1,$$

where $V_*(x) = 1 - \overline{V}_*(x)$. Let

$$\xi_i^* = V_*^{-1}(V(\xi_i)), \quad i \geq 1. \tag{3.23}$$

It is easy to verify that $\{\xi_i^*, i \geq 1\}$ are still WUOD and identically distributed r.v.s with common distribution $V_* \in \mathcal{C}$. By the definition of $\overline{V}_*$, we know that $\xi_i \leq \xi_i^*$, $i \geq 1$, and then $a \leq E\xi_i^* = a^* < \infty$. Thus, $K_1 + K_2$ in (3.10) is divided into three parts as

$$K_1 + K_2 \leq \left(\sum_{n=1}^{m_0} + \sum_{n=m_0+1}^{(1-\delta)x/a^*} \right) P\left(\sum_{i=1}^{n} \xi_i^* > x \right) P(\eta = n)$$

$$+ \sum_{(1-\delta)x/a^* < n \leq (1-\delta)x/a} P\left(\sum_{i=1}^{n} \xi_i > x \right) P(\eta = n) \tag{3.24}$$

$$= K_1' + K_2' + K_2''.$$

Clearly, $\{\xi_i^*, i \geq 1\}$ are such that the conditions of Lemmas 3.2 and 3.3 hold, then (3.11) and (3.13) can still hold with K_1, K_2, and $\{\xi_i, i \geq 1\}$ replaced by K_1', K_2', and $\{\xi_i^*, i \geq 1\}$. So, we deduce by $V_* \in \mathcal{C} \subset \mathfrak{D}$ and $\overline{V}_*(x) = o(\overline{U}(x))$ that

$$K_1' + K_2' \lesssim (b + \varepsilon) \cdot \frac{\overline{V}_*(x)}{\overline{V}_*(x/a)} \cdot \frac{\overline{V}_*(x/a)}{\overline{U}(x/a)} \cdot \overline{U}\left(\frac{x}{a}\right) = o\left(\overline{U}\left(\frac{x}{a}\right)\right). \tag{3.25}$$

For K_2'', it follows from Lemma 3.4 that

$$\lim_{n \to \infty} P\left(\frac{\sum_{i=1}^{n} \xi_i}{n} - a > \frac{a\delta}{1 - \delta} \right) = 0. \tag{3.26}$$

Then, by $U \in \mathcal{C} \subset \mathfrak{D}$ we have

$$K_2'' \leq \sum_{(1-\delta)x/a^* < n \leq (1-\delta)x/a} P\left(\frac{\sum_{i=1}^{n} \xi_i}{n} - a > \frac{a\delta}{1 - \delta} \right) P(\eta = n)$$

$$= o\left(\overline{U}\left(\frac{(1-\delta)x}{a^*}\right)\right) = o\left(\overline{U}\left(\frac{x}{a}\right)\right). \tag{3.27}$$

From (3.10), (3.14), and (3.24)–(3.27), we find that

$$P\left(\sum_{i=1}^{\eta}\xi_i > x\right) \lesssim \overline{U}\left(\frac{x}{a}\right).$$ (3.28)

Again by (3.16) and (3.19), it is seen that

$$P\left(\sum_{i=1}^{\eta}\xi_i > x\right) \geq K_4 \gtrsim \overline{U}\left(\frac{x}{a}\right).$$ (3.29)

Since $U \in \mathcal{C} \subset \mathfrak{D}$ and $\overline{V}(x) = o(\overline{U}(x))$, we get

$$\overline{V}(x) = \frac{\overline{V}(x)}{\overline{U}(x)} \cdot \frac{\overline{U}(x)}{\overline{U}(x/a)} \cdot \overline{U}\left(\frac{x}{a}\right) = o\left(\overline{U}\left(\frac{x}{a}\right)\right).$$ (3.30)

Consequently, we obtain by combining (3.28)–(3.30) that (3.8) holds for $c = 0$.

(2) Now we deal with the case that $c = \infty$, namely, $\overline{U}(x) = o(\overline{V}(x))$. Apparently, when $V \in \mathcal{L} \cap \mathfrak{D}$, we can derive by Lemmas 3.2 and 3.3 that (3.11) and (3.13) still hold. As for K_3, by $\overline{U}(x) = o(\overline{V}(x))$ and $V \in \mathcal{L} \cap \mathfrak{D}$, we know that

$$K_3 \leq \frac{\overline{U}((1-\delta)x/a)}{\overline{V}((1-\delta)x/a)} \cdot \frac{\overline{V}((1-\delta)x/a)}{\overline{V}(x)} \cdot \overline{V}(x) = o\left(\overline{V}(x)\right).$$ (3.31)

Substituting (3.11), (3.13), and (3.31) into (3.10) implies that

$$P\left(\sum_{i=1}^{\eta}\xi_i > x\right) \lesssim b\overline{V}(x).$$ (3.32)

For $V \in \mathcal{L} \cap \mathfrak{D}$, by Lemma 3.2 we also get (3.17). As for K_4, arguing as (3.19) and (3.31), we still have that

$$K_4 \gtrsim \overline{U}\left(\frac{(1+\delta)x}{a}\right) = o\left(\overline{V}(x)\right).$$ (3.33)

From (3.16), (3.17), and (3.33), we conclude that

$$P\left(\sum_{i=1}^{\eta}\xi_i > x\right) \gtrsim b\overline{V}(x).$$ (3.34)

Similarly to the derivation of (3.30), by $\overline{U}(x) = o(\overline{V}(x))$ and $V \in \mathcal{L} \cap \mathfrak{D}$ we still see that $\overline{U}(x/a) = o(b\overline{V}(x))$. This, along with (3.32) and (3.34), gives relation (3.8) immediately.

(3) According to the proof of (2), we know that if $V \in \mathcal{L} \cap \mathfrak{D}$, then (3.11) and (3.13) hold. While from the proof of (1), we have (3.14) if $U \in \mathcal{C}$. Hence, under the conditions of (3), we obtain (3.15).

On the other hand, from the proof of (2), we can get (3.17) when $V \in \mathcal{L} \cap \mathfrak{D}$. Again by the proof of (1), relation (3.19) also holds for $U \in \mathcal{C}$. So, (3.20) is proved under the conditions of (3). As a result, we show (3.8) directly. $\qquad\square$

The next two lemmas will give some results of the renewal risk model, which is the compound renewal risk model with $N_1 = N_2 = \cdots = 1$.

Condition 5. For α in (2.5), there exist $t_0 \in \Lambda \cap (0, \infty)$ and f in (1.12) such that $(f(x))^{\alpha}(P(\theta_1 \leq t_0))^{f(x)} = o(\overline{F}(x))$.

Lemma 3.6 (Corollary 2.1 of Wang et al. [9]). *Consider the compound renewal risk model with $N_1 = N_2 = \cdots = 1$ and $c > \lambda\mu$, in which $\{X_i, i \geq 1\}$ are i.i.d. r.v.s with common distribution F, and $\{\theta_i, i \geq 1\}$ are WLOD r.v.s satisfying (2.5) and one of Conditions 1–4.*

(1) If $F \in \mathcal{S}_*$, then for any $t_0 \in \Lambda$, it holds uniformly for all $t \in (t_0, \infty)$ that

$$\psi(x, t) \sim \frac{\lambda}{c - \lambda\mu} \int_x^{x+\mu\lambda(t)} \overline{F}(s)ds. \tag{3.35}$$

(2) Furthermore, if Condition 5 holds, then relation (3.35) still holds uniformly for all $t \in \Lambda$.

Lemma 3.7 (Corollary 2.2 of Wang et al. [9]). *Under conditions of Lemma 3.6 and assumption H_4, one has*

(1) if $F \in \mathcal{S}_*$, then

$$\psi(x, \tau) \sim \frac{\lambda}{c - \lambda\mu} E \int_x^{x+\mu\lambda(\tau)} \overline{F}(s)ds \tag{3.36}$$

holds uniformly for all $\tau \in \{\tau : P(\tau I_{\{\tau \geq t_0\}} \in \Lambda) > 0\}$ for any $t_0 \in \Lambda$.

(2) Additionally, if Condition 5 holds, then relation (3.36) still holds uniformly for all $t \in \Delta$.

Now we prove the main results as follows.

Proof of Theorem 2.1. Clearly, if $F \in \mathcal{L} \cap \mathfrak{D}$ then F satisfies Condition 5. In fact, the assumption $F \in \mathcal{L} \cap \mathfrak{D}$ indicates that the tail of F behaves essentially like a power function, thus there exists $q > 0$ such that $x^q \overline{F}(x) \to \infty$. Take $f(x) = x^p$ for any $0 < p < 1$, which satisfies (1.12). So, for any $s > 0$, $e^{-sx^p}/\overline{F}(x) \to 0$ and Condition 5 holds. $\qquad\square$

First, consider Theorem 2.1(1). For the case that $0 \leq c < \infty$, we can obtain relation (2.4) by (1.5) and Lemmas 3.5(1) and 3.6, and going along the similar ways to that of

Theorems 2.2(2) and (3) of Aleškevičienė et al. [2]. For the case that $c = \infty$, since $F \in \mathcal{L} \cap \mathfrak{D}$, by Lemma 3.5(2) we have that for any $y > 0$,

$$\limsup_{x \to \infty} \frac{P(S_N > xy)}{P(S_N > x)} = \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty, \qquad \lim_{x \to \infty} \frac{P(S_N > x + y)}{P(S_N > x)} = \lim_{x \to \infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1,$$

(3.37)

which tell us that the distribution of S_N belongs to the class $\mathcal{L} \cap \mathfrak{D}$. Then, arguing as the proof of Theorem 2.2(1) of Aleškevičienė et al. [2], we also obtain relation (2.4).

Now consider Theorem 2.1(2). By Lemma 3.5(3), $F \in \mathcal{L} \cap \mathfrak{D}$ and $G \in \mathcal{C}$, it is also clear that the distribution of S_N belongs to $\mathcal{L} \cap \mathfrak{D}$. Hence, relation (2.4) still holds from (1.5) and Lemmas 3.5(3) and 3.6.

Proof of Theorem 2.2. By (1.6), the uniformity of (2.4) in Theorem 2.1, and the independence between τ and the risk system, we can get the proof of Theorem 2.2. $\qquad\square$

Acknowledgments

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