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Option Pricing Under an Abnormal Economy: using the Square Root of the Brownian Motion

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Abstract:

Purpose: The literature on option pricing is typically suitable to usual circumstances (normal economy). However, in general, under unusual economic states, the traditional models of options are not suitable. Therefore, there is a need to consider alternative stochastic processes and models that captures the unusual states of the economy.

Design/methodology/approach: In this connection, we bridge the gap in the literature by providing a simple, explicit pricing formula for the European option under both normal and abnormal economies.

Findings: In this paper, we first discuss the background theory for the Black-Scholes model under a normal economy when there are no unusual changes in the price of the underlying so that Brownian motion works well. We then provide a simple, explicit pricing formula for the European option under both normal and abnormal economies. This formula is as simple as the classical Black-Scholes formula and there is no need for computational methods. In doing so, we utilize a nontraditional process (the square root of the Brownian motion) and complex analysis. We also rely on a non-traditional stochastic process. Thereafter, we construct three examples to illustrate the use of our proposed model.

Originality/Value:

Practical implications: The theory developed in this paper is used for investors for their investments and is useful for policy-makers in setting up some rules for the options markets.

Keywords: Option pricing, stochastic volatility, jump, abnormal economy, Brownian Motion

JEL classifications: G0

1 Introduction

Nowadays, there is an urgent need for a convenient option price formula in an abnormal economy like strong/exceptional recessions or booms, such as the economy in wars or pandemics. To account for unusual changes in asset prices in an abnormal economy, some studies adopt jump diffusions. However, these studies do not offer explicit (simple) formulas for the price of the European option in an abnormal economy. Instead, they rely on numerical or computational methods. Similarly, other literature on stochastic volatility such as Hull and White (1987), Chen *et al.* (2016), Gong and Zhuang (2016) and Kleinert and Korbel (2016) does not provide explicit formulas in an abnormal economy. On the other hand, Leippold and Schärer (2017), Zhang and Wang (2013) and Zhang *et al.* (2012) provide empirical analysis and other studies used a numerical or a computational approach. For example, Zhou *et al.* (2013) and Martino *et al.* (2015). Alghalith (2020, 2021) used a different process and a different method. Guo *et al.* (2021) used a time driven approach.

Studies that dealt with option pricing under a stochastic interest rate rely on Monte Carlo simulations, finite difference and/or Fourier transforms. For example, Heston (1993) and He and Zhu (2018) rely on fast Fourier transforms and Sun and Xu (2018) use Monte Carlo methods. Heston (1993) obtained a closed-form solution for the price of a European call option on an asset with stochastic volatility by allowing an arbitrary correlation between volatility and returns. However, our solution is different from that obtained by Heston (1993). Heston's formula/model is far more complex. It requires the use of Fast Fourier Transforms. It is inaccurate, especially for short options. The parameters need to be carefully calibrated. It fails to adequately capture the implied volatility. Our method solves these limitations.

With regard to an unusual economy, Balcereka *et al.* (2022) consider a model that can be used during a recession. In doing so, they adopt the subordinated Cox-Ross-Rubinstein model. In this paper, we use a different approach to model the normal/abnormal states of the economy as a (smooth) switching regime in the same stochastic equations. In doing, so we provide simple, explicit pricing formulas for the European option under both normal and abnormal economies.

2 Background Theory

The Black-Scholes model can be used under a normal economy when there are no unusual changes in the price of the underlying so that Brownian motion works well (Merton, 1976). Under the Black-Scholes model, the dynamics of the underlying asset are given by

$$dS_u = S_u \left[r du + \sigma dW(u) \right] \ , \tag{2.1}$$

where r is the constant interest rate, W is a Brownian motion, u is time, and σ is the constant volatility. Thus, the dynamics of the call option price is given by

$$dC(u, S) = C_u du + C_S dS + \frac{1}{2} C_{SS} (dS)^2 \ , \tag{2.2}$$

and

$$C_t dt + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC = 0 \ . \tag{2.3}$$

As a result, price of the option is

$$C(t, S) = SN(d_1) - e^{-r(T-t)} KN(d_2) \ , \tag{2.4}$$

where $d_1 = \frac{\ln(S/K)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$, $d_2 = d_1 - \sqrt{\sigma^2 (T-t)}$, K is the strike price and T is the expiration date and t is the current time.

We summarize the above results into the following proposition:

Proposition 2.1 *Under a normal economy, the price of the call option is*

$$C(t, Y_t) = Y_t N(d_1) - e^{-r(T-t)} K N(d_2), \tag{2.5}$$

where $d_1 = \frac{\ln(Y_t/K)+\left(r+\sigma^2/2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$ *and* $d_2 = d_1 - \sqrt{\sigma^2 (T-t)}$, *$r$ is the constant interest rate, W is a Brownian motion, u is time, and σ is the constant volatility.*

To model option price under an normal economy, Heston (1993) uses the stochastic volatility. However, Heston's formula/model is far more complex. It requires the use of Fast Fourier Transforms. It is inaccurate, especially for short options. The parameters need to be carefully calibrated. It fails to adequately capture the implied volatility. Our paper contributes to the liteature by solving these limitations. We discuss our contributions to the literature more in the next section.

3 Our Proposed Model

The limitation of the traditional approach to estimating the price of options is that it can only be used for the option price in a normal economy. To bridge the gap in the literature to overcome the limitations of the traditional approach, in this paper, we develop a theory to model the price of options under both a normal economy and an abnormal economy. We develop the theory in this section. We define an abnormal economy as a strong/exceptional recession or boom, such as the economy in wars or pandemics. A normal economy is modeled by a Brownian motion. The abnormal economy is modeled by a different stochastic process.

The square root Brownian motion (Frasca *et al.,* 2021; Frasca, 2017; Frasca and Farina, 2017) is suitable for modeling abnormal states because its variance is $O\left(\sqrt{du}\right)$ which is more volatile compared to a Brownian motion (BM). Also, its mean is not zero. The key idea is to model the normal/abnormal states as a (smooth) switching regime in the same equations. The stochastic process $x(u)$ controls the switch based on an economic thresh old or indicator.

By applying the square root Brownian motion, we improve Proposition 2.1 into the following theorem:

Theorem 3.1 *The price of the European call is given by*

$$C(t,S) = SN(d_1) - e^{-r(T-t)}KN(d_2) \ , \tag{3.6}$$

where $d_1 = \frac{\ln(S/K)+\left(r+\frac{\delta}{2}\right)(T-t)}{\sqrt{\delta(T-t)}}$ *and* $d_2 = d_1 - \sqrt{\delta(T-t)}$.

Proof. The dynamics of the stock price are given by

$$\begin{aligned} dS(u) &= -S(u)\frac{x(u)}{y(u)}\left[rdu + \sigma dW_1(u)\right] + \qquad (3.7) \\ & S(u)\left(1+\frac{x(u)}{y(u)}\right)\left[\begin{array}{c} (r+\varphi(u)v(u))\,du + \theta(u)\sqrt{dW_2(u)} + \\ \Psi(u)\,z(u)\sqrt{dW_3(u)} \end{array}\right], \end{aligned}$$

where $v = 1/\sqrt{du}$, $E\varphi(u) = 0$, $E\theta(u) = 0$, $E\Psi(u) = 0$, $z(u) = \phi(u)/c(du)^{1.25}$, $y \neq 0$, $y(u) = -x(u)$ are random; all the processes are independent. We note that θ and Ψ are transform functions in the sense of Alghalith (2020) to ensure that $dS(u)$ is real. The power 1.25 is used to show that $dS(u)/S(u)$ goes to infinity (a jump).

Under a normal state, $x(u) \neq 0$ and thus the second term is zero; therefore, the classical

Black-Scholes equation holds. Under an abnormal state $x(u) = 0$, thus the first term is zero. The term $\Psi z(u)\sqrt{dW_3(u)}$ is a jump in the sense that $\Psi z(u)\sqrt{dW_3(u)}/du$ (the slope) tends to infinity. In the abnormal state, we can show that the price of the option is an adjustment to the Black-Scholes formula, where σ^2 is replaced by $E\varphi(u)^2 \equiv \delta$ (the abnormal state is much more volatile). To show this, the dynamics of the European call are

$$dC(u,S) = C_u du + C_S dS + \frac{1}{2}S^2 C_{SS}(dS)^2 \ . \tag{3.8}$$

Substituting (3.7) into (3.8) yields

$$\begin{aligned} dC(u,S) &= C_u du + C_S \left[\begin{array}{c} -S(u)\frac{x(u)}{y(u)}\left[rdu + \sigma dW_1(u)\right] + \\ S(u)\left(1+\frac{x(u)}{y(u)}\right)\left[\begin{array}{c} (r+\alpha v)\,du + \theta(u)\sqrt{dW_2(u)} + \\ \Psi(u)\,z(u)\sqrt{dW_3(u)} \end{array}\right] \end{array}\right] + \\ & \frac{1}{2}S^2 C_{SS} \left\{ \begin{array}{c} - \\ \frac{x(u)}{y(u)}\left[rdu + \sigma dW_1(u)\right] + \\ \left(1+\frac{x(u)}{y(u)}\right)\left[\begin{array}{c} (r+\alpha v)\,du + \theta(u)\sqrt{dW_2(u)} + \\ \Psi(u)\,z(u)\sqrt{dW_3(u)} \end{array}\right] \end{array}\right\}^2 \ . \end{aligned} \tag{3.9}$$

Recognizing that, under risk neutrality, $E_t dC(t,S) = rC(t,S)\,dt$, we obtain

$$C_t dt + rSC_S + \frac{1}{2}\delta S^2 C_{SS} - rC = 0 \ , \tag{3.10}$$

where $\delta \equiv E\varphi(u)^2$ under abnormal economy and $\delta \equiv \sigma^2$ under abnormal economy. However, the switch between $E\varphi(u)^2$ and σ^2 is smooth. That is, there is no need to have two distinct partial differential equations and a boundary problem.

Clearly, the option price in the abnormal state is

$$C(t,S) = SN(d_1) - e^{-r(T-t)}KN(d_2) \ , \tag{3.11}$$

where $d_1 = \frac{\ln(S/K)+\left(r+\frac{\delta}{2}\right)(T-t)}{\sqrt{\delta(T-t)}}$ and $d_2 = d_1 - \sqrt{\delta\,(T-t)}$.

$\square$

We note that δ can be estimated in a similar way as σ^2. Or it can be calibrated. Moreover, the implied value of δ can be estimated, and then the implied value can be used to estimate δ. In addition, we make the following remark:

Remark 3.1 *The advantage of Theorem 3.1 over Proposition 2.1 is that it allows for stochastic volatility, abnormal processes/economies as well as normal processes/economies.*

4 Illustrations

In this section, we illustrate the usefulness of our proposed model by providing three simple and practical examples that allow for stochastic volatility, abnormal processes/economies as well as normal processes/economies while the traditional approach can only work for normal processes/economies.

. We show that the calculation of the implied volatility is straightforward by using the first example. We suggest how the volatility parameter can be estimated by using the second example and show an alternative way of estimating the volatility parameter. We first illustrate the first example as follows:

Example 4.1 *If the market price of the option (under an abnormal economy)* $C_M = 30$, $r = .03$, $S = K = 100$, $T - t = 1$, *then* $\hat{\delta}$ *(the implied value of* δ*)* $= .742$.

It is clear that in Example 4.1, the calculation of the implied volatility is straightforward. The calculation of the implied volatility is easy as the one in the Black-Scholes model.

Example 4.2 *Using the above example, calculate $\hat{\delta}$ for each previous day since the start of the abnormal state, then use the average of these values as an estimate of δ. The estimated value of δ will be used to price the option using our formula.*

Example 4.2 tells us how to estimate the volatility parameter. Estimating the volatility parameter can be done using the methods used in the classical Black-Scholes model.

Example 4.3 *Let R_i be the realizations of the returns of the underlying asset during the abnormal state, then δ can be estimated as $\frac{\sum_i R_i^2}{n}$, where n is number of observations.*

Example 4.3 shows an alternative way of estimating the volatility parameter. The advantage of applying Theorem 3.1 over Proposition 2.1 in this example is its simplicity. We remark again that Examples 4.1 to 4.3 allow for stochastic volatility, abnormal processes/economies as well as normal processes/economies while the traditional approach can only work for normal processes/economies.

5 Discussions and Concluding remarks

In this paper, we extend Alghalith and Wong (2022) and others by providing a single formula that simultaneously works for both abnormal and normal economies. This formula is as simple as the classical Black-Scholes formula and there is no need for computational methods. In doing so, we utilize a nontraditional process (the square root of the Brownian motion) and complex analysis.

The literature on option pricing is typically suitable to usual circumstances (normal economy). However, in general, under unusual economic states, the traditional models of options are not suitable. Therefore, there is a need to consider alternative stochastic processes and models that captures the unusual states of the economy. This is especially true nowadays given the abnormal state of the global economy and local economies. Furthermore, our model is applicable to both normal and abnormal states. Key advantages of our model are its simplicity and flexibility. Consequently, this model will be appealing to academics, practitioners, and policymakers. However, the limitation of the model is that the estimation of the volatility parameters is more involved than estimating the standard deviation (used for the Black-Scholes model).

Consequently, our paper provides the foundations for numerous future studies that will deal with abnormal economic states. Future researchers can also apply our approach to other derivatives, such as American and Asian options..

Future researchers can also apply our approach to other derivatives, such as American and Asian options. It also can be applied to other areas in finance, such as the portfolio models.

The approach developed in our paper can be used to get better estimations of option prices for some of the important issues in options, see, for example, Abid, *et al.* (2009) and Wong, *et al.* (2011). Academics and practitioners could also apply our approach to get better estimates of other financial products, for example, bond (Edeki, *et al.,* 2021; Guimarães, *et al.,* 2021; Ravinagarajan and Sophia, 2022), stock (Obrimah and Wong, 2022; Ma and Wang, 2021; Yeung and Wong, 2022), warrant (Chan, *et al.,* 2012), futures (Lean, *et al.,* 2010), and others. There are many other issues that one can apply our approach to analyze. One may read Alghalith, *et al.* (2021) and Wong (2020, 2021) for more issues. Moreover, getting better estimations of the options prices could be useful in helping investors make better decisions in their investment (Wong, 2007) and help policymakers make better decisions in their policy-making (Tiwari, *et al.,* 2021; Saungweme, *et al.,* 2021; Aye, 2021).

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